Section 8.1
Matrix Solutions to Linear Systems

Avoiding the Scale?

We have already studied guidelines, given as a set of linear inequalities, that tell how much we should weigh based on age and height. But what do people really weigh? The chart below shows average weights of American adults by age and gender.

In this section of the textbook, we will look at how such data can be placed in something called a matrix and how matrices can be used to solve linear systems.

	Ages 20–29	Ages 30–39	Ages 40–49	Ages 50–59	Ages 60–69	Ages 70–79	Ages 80+
Men	188	194	202	199	198	187	168
Women	156	165	171	172	171	156	142

Source: National Center for Health Statistics

Objective #1: Write the augmented matrix for a linear system.

✔ *Solved Problem #1*

1. Write the augmented matrix for the system:

$$\begin{cases} 3x + 4y = 19 \\ 2y + 3z = 8 \\ 4x - 5z = 7 \end{cases}$$

The augmented matrix has a row for each equation and a vertical bar separating the coefficients of the variables on the left from the constants on the right. Coefficients of the same variable are lined up vertically in the same column. If a variable is missing from an equation, its coefficient is 0.

It may be helpful to view the system as

$$\begin{cases} 3x + 4y + 0z = 19 \\ 0x + 2y + 3z = 8 \\ 4x + 0y - 5z = 7 \end{cases}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 4 & 0 & 19 \\ 0 & 2 & 3 & 8 \\ 4 & 0 & -5 & 7 \end{array}\right]$$

✎ *Pencil Problem #1* ✎

1. Write the augmented matrix for the system:

$$\begin{cases} 5x - 2y - 3z = 0 \\ x + y = 5 \\ 2x - 3z = 4 \end{cases}$$

Objective #2: Perform matrix row operations.

✔ *Solved Problem #2*

2a. Perform the row operation and write the new matrix:

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right];\ \frac{1}{4}R_1$$

Multiply each element in row 1 by $\frac{1}{4}$. The elements in row 2 and row 3 do not change.

$$\left[\begin{array}{ccc|c} \frac{1}{4}(4) & \frac{1}{4}(12) & \frac{1}{4}(-20) & \frac{1}{4}(8) \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 3 & -5 & 2 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right]$$

2b. Perform the row operation and write the new matrix:

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ -3 & -2 & 1 & -9 \end{array}\right];\ 3R_2 + R_3$$

Multiply each element in row 2 by 3 and add to the corresponding element in row 3. Replace the elements in row 3. Row 1 and row 2 do not change.

$$\left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 3(1)+(-3) & 3(6)+(-2) & 3(-3)+1 & 3(7)+(-9) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 4 & 12 & -20 & 8 \\ 1 & 6 & -3 & 7 \\ 0 & 16 & -8 & 12 \end{array}\right]$$

✎ *Pencil Problem #2* ✎

2a. Perform the row operation and write the new matrix:

$$\left[\begin{array}{ccc|c} 2 & -6 & 4 & 10 \\ 1 & 5 & -5 & 0 \\ 3 & 0 & 4 & 7 \end{array}\right];\ \frac{1}{2}R_1$$

2b. Perform the row operation and write the new matrix:

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 3 & 1 & -1 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right];\ -3R_1 + R_2$$

Objective #3: Use matrices and Gaussian elimination to solve systems.

✔ *Solved Problem #3*

3. Use matrices to solve the system: $\begin{cases} 2x+y+2z=18 \\ x-y+2z=9 \\ x+2y-z=6 \end{cases}$

Write the augmented matrix for the system.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 18 \\ 1 & -1 & 2 & 9 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

We want a 1 in the upper left position. One way to do this is to interchange row 1 and row 2.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 18 \\ 1 & -1 & 2 & 9 \\ 1 & 2 & -1 & 6 \end{array}\right] R_1 \leftrightarrow R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 2 & 18 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

Now we want zeros below the 1 in the first column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 2 & 18 \\ 1 & 2 & -1 & 6 \end{array}\right] -2R_1 + R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 1 & 2 & -1 & 6 \end{array}\right] -R_1 + R_3 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

Next we want a 1 in the second row, second column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -3 & -3 \end{array}\right] \frac{1}{3}R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

Now we want a zero below the 1 in the second row, second column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 3 & -3 & -3 \end{array}\right] -3R_2 + R_3 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

✎ *Pencil Problem #3* ✐

3. Use matrices to solve: $\begin{cases} x+y-z=-2 \\ 2x-y+z=5 \\ -x+2y+2z=1 \end{cases}$

Next we want a 1 in the third row, third column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & -1 & -3 \end{array}\right] \quad -R_3 = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The resulting system is:

$$\begin{aligned} x - y + 2z &= 9 \\ y - \frac{2}{3}z &= 0 \\ z &= 3 \end{aligned}$$

Back-substitute 3 for z in the second equation.

$$\begin{aligned} y - \frac{2}{3}(3) &= 0 \\ y - 2 &= 0 \\ y &= 2 \end{aligned}$$

Back-substitute 2 for y and 3 for z in the first equation.

$$\begin{aligned} x - y + 2z &= 9 \\ x - (2) + 2(3) &= 9 \\ x - 2 + 6 &= 9 \\ x + 4 &= 9 \\ x &= 5 \end{aligned}$$

$(5,2,3)$ satisfies both equations.

The solution set is $\{(5,2,3)\}$.

Objective #4: Use matrices and Gauss-Jordan elimination to solve systems.

✔ Solved Problem #4

Solve the system by Gauss-Jordan elimination. Begin with the matrix obtained in Solved Problem #3.

$$\begin{cases} 2x + y + 2z = 18 \\ x - y + 2z = 9 \\ x + 2y - z = 6 \end{cases}$$

The final matrix from Solved Problem #3 is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

First we want a zero above the 1 in the second column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R_2 + R_1 = \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Now we want zeros above the 1 in the third column.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 9 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad -\frac{4}{3}R_3 + R_1 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \frac{2}{3}R_3 + R_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This last matrix corresponds to

$$x = 5,\ y = 2,\ z = 3$$

The solution set is $\{(5, 2, 3)\}$.

✎ Pencil Problem #4 ✎

4a. Solve the system by Gauss-Jordan elimination. Begin with the matrix obtained in Pencil Problem #3.

$$\begin{cases} x + y - z = -2 \\ 2x - y + z = 5 \\ -x + 2y + 2z = 1 \end{cases}$$

<u>Answers</u> for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\left[\begin{array}{rrr|r} 5 & -2 & -3 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 4 \end{array}\right]$ *(8.1 #5)*

2a. $\left[\begin{array}{rrr|r} 1 & -3 & 2 & 5 \\ 1 & 5 & -5 & 0 \\ 3 & 0 & 4 & 7 \end{array}\right]$ *(8.1 #13)* **2b.** $\left[\begin{array}{rrr|r} 1 & -3 & 2 & 0 \\ 0 & 10 & -7 & 7 \\ 2 & -2 & 1 & 3 \end{array}\right]$ *(8.1 #15)*

3. $\{(1, -1, 2)\}$ *(8.1 #21)*

4. $\{(1, -1, 2)\}$ *(8.1 #21)*

Section 8.2
Inconsistent and Dependent Systems and Their Applications

Lane Closed Ahead!
Be Prepared to Stop!

You've allowed yourself barely enough time to get to campus, and now you see a sign for road construction along your normal route. Should you take an alternate route to make it to class on time?
In this section of the textbook, we use systems of equations to model traffic flow. Systems of equations with more than one solution can tell us how many cars should be directed toward alternate routes when flow along one street is limited by road work.

Objective #1: Apply Gaussian elimination to systems without unique solutions.

✔ *Solved Problem #1*

1a. Use Gaussian elimination to solve the system:

$$\begin{cases} x - 2y - z = -5 \\ 2x - 3y - z = 0 \\ 3x - 4y - z = 1 \end{cases}$$

Write the augmented matrix for the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 2 & -3 & -1 & 0 \\ 3 & -4 & -1 & 1 \end{array}\right]$$

Attempt to simplify the matrix to row-echelon form. The matrix already has a 1 in the upper left position. We want 0s below the 1. Multiply row 1 by −2 and add to row 2, and multiply row 1 by −3 and add to row 3.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 10 \\ 0 & 2 & 2 & 16 \end{array}\right]$$

Now we want a 0 below the 1 in the second column. Multiply row 2 by −2 and add to row 3.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

The last row represents $0x + 0y + 0z = -4$, which is false. The system has no solution. The solution set is $\varnothing$, the empty set.

✎ *Pencil Problem #1* ✐

1a. Use Gaussian elimination to solve the system:

$$\begin{cases} 5x + 12y + z = 10 \\ 2x + 5y + 2z = -1 \\ x + 2y - 3z = 5 \end{cases}$$

1b. Use Gaussian elimination to solve the system:

$$\begin{cases} x - 2y - z = 5 \\ 2x - 5y + 3z = 6 \\ x - 3y + 4z = 1 \end{cases}$$

We begin with the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 2 & -5 & 3 & 6 \\ 1 & -3 & 4 & 1 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & -1 & 5 & -4 \\ 0 & -1 & 5 & -4 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & 1 & -5 & 4 \\ 0 & -1 & 5 & -4 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 5 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The original system is equivalent to the system

$$\begin{cases} x - 2y - z = 5 \\ y - 5z = 4 \end{cases}$$

The system is consistent and the equations are dependent. Express x and y in terms of z.

$$y - 5z = 4$$
$$y = 5z + 4$$

$$x - 2y - z = 5$$
$$x = 2y + z + 5$$
$$x = 2(5z + 4) + z + 5$$
$$x = 10z + 8 + z + 5$$
$$x = 11z + 13$$

Each ordered pair of the form $(11z + 13, 5z + 4, z)$ is a solution of the system. The solution set is $\{(11z + 13, 5z + 4, z)\}$.

1b. Use Gaussian elimination to solve the system:

$$\begin{cases} 8x + 5y + 11z = 30 \\ -x - 4y + 2z = 3 \\ 2x - y + 5z = 12 \end{cases}$$

Objective #2: Apply Gaussian elimination to systems with more variables than equations.

✔ *Solved Problem #2*

2. Use Gaussian elimination to solve the system:

$$\begin{cases} x + 2y + 3z = 70 \\ x + y + z = 60 \end{cases}$$

We begin with the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 70 \\ 1 & 1 & 1 & 60 \end{array}\right]$$

$$\xrightarrow{-R_1+R_2}\left[\begin{array}{ccc|c} 1 & 2 & 3 & 70 \\ 0 & -1 & -2 & -10 \end{array}\right]$$

$$\xrightarrow{-R_2}\left[\begin{array}{ccc|c} 1 & 2 & 3 & 70 \\ 0 & 1 & 2 & 10 \end{array}\right]$$

The original system is equivalent to the system

$$\begin{cases} x + 2y + 3z = 70 \\ y + 2z = 10 \end{cases}$$

Express x and y in terms of z.

$$y + 2z = 10$$
$$y = -2z + 10$$
$$x + 2y + 3z = 70$$
$$x = -2y - 3z + 70$$
$$x = -2(-2z + 10) - 3z + 70$$
$$x = 4z - 20 - 3z + 70$$
$$x = z + 50$$

Each ordered pair of the form $(z + 50, -2z + 10, z)$ is a solution of the system. The solution set is $\{(z + 50, -2z + 10, z)\}$.

✎ *Pencil Problem #2* ✎

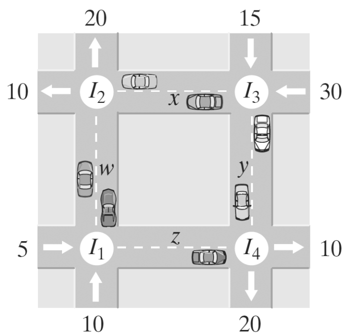
2. Use Gaussian elimination to solve the system:

$$\begin{cases} 2x + y - z = 2 \\ 3x + 3y - 2z = 3 \end{cases}$$

Objective #3: Solve problems involving systems without unique solutions.

✔ Solved Problem #3

3. The figure shows a system of four one-way streets. The numbers in the figure denote the number of cars per minute that travel in the direction shown.



3a. Use the requirement that the number of cars entering each of the intersections per minute must equal the number of cars leaving per minute to set up a system of equations that keeps traffic moving.

Consider one intersection at a time.

I_1: 10 + 5 = 15 cars enter and $w + z$ leave, so $w + z = 15$.

I_2: $w + x$ cars enter and 10 + 20 = 30 leave, so $w + x = 30$.

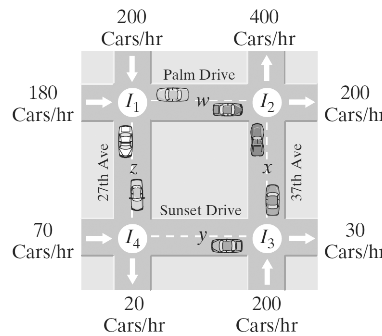
I_3: 15 + 30 = 45 cars enter and $x + y$ leave, so $x + y = 45$.

I_4: $y + z$ cars enter and 20 + 10 = 30 leave, so $y + z = 30$.

The system is $\begin{cases} w+z=15 \\ w+x=30 \\ x+y=45 \\ y+z=30 \end{cases}$

✎ Pencil Problem #3 ✎

3. The figure shows a system of four one-way streets. The numbers in the figure denote the number of cars per hour that travel in the direction shown.



3a. Use the requirement that the number of cars entering each of the intersections per hour must equal the number of cars leaving per hour to set up a system of equations that keeps traffic moving.

3b. Use Gaussian elimination to solve the system.

We begin with the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 1 & 1 & 0 & 0 & 30 \\ 0 & 1 & 1 & 0 & 45 \\ 0 & 0 & 1 & 1 & 30 \end{array}\right]$$

$$\xrightarrow{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & -1 & 15 \\ 0 & 1 & 1 & 0 & 45 \\ 0 & 0 & 1 & 1 & 30 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & -1 & 15 \\ 0 & 0 & 1 & 1 & 30 \\ 0 & 0 & 1 & 1 & 30 \end{array}\right]$$

$$\xrightarrow{-R_3+R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & -1 & 15 \\ 0 & 0 & 1 & 1 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From the first row, we get $w + z = 15$, so $w = 15 - z$. From the second and third rows, we get $x - z = 15$ and $y + z = 30$, respectively, so $x = 15 + z$ and $y = 30 - z$. The solution set is $\{(15 - z, 15 + z, 30 - z, z)\}$.

3b. Use Gaussian elimination to solve the system.

3c. If construction limits z to 10 cars per minute, how many cars per minute must pass between the other intersections to keep traffic flowing?

Substitute 10 for z in the system's solution.

$(15 - z, 15 + z, 30 - z, z)$

$= (15 - 10, 15 + 10, 30 - 10, 10)$

$= (5, 25, 20, 10)$

To keep traffic flowing, we must have $w = 5$, $x = 25$, and $y = 20$ cars per minute.

3c. If construction limits z to 50 cars per hour, how many cars per hour must pass between the other intersections to keep traffic flowing?

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. no solution or $\varnothing$ *(8.2 #1)* **1b.** $\{(5-2z, -2+z, z)\}$ *(8.2 #7)*

2. $\left\{\left(1+\frac{1}{3}z,\ \frac{1}{3}z,\ z\right)\right\}$ *(8.2 #15)*

3a. $\begin{cases} w+z=380 \\ w+x=600 \\ x-y=170 \\ y-z=50 \end{cases}$ *(8.2 #33a)* **3b.** $\{(380-z, 220+z, 50+z, z)\}$ *(8.2 #33b)*

3c. $w = 330$, $x = 270$, $y = 100$ *(8.2 #33c)*

Section 8.3
Matrix Operations and Their Applications

Making Things Clearer

Have you ever had trouble reading a document because the text didn't differ sufficiently from the background? By increasing the contrast between the text and the background, you can often make the document easier to read.

In this section of the textbook, we use matrix operations to change the contrast between a letter and its background and to transform figures through translations, stretching or shrinking, and reflections.

Objective #1: Use matrix notation.

✔ *Solved Problem #1*

1. Let $A = \begin{bmatrix} 5 & -2 \\ -3 & \pi \\ 1 & 6 \end{bmatrix}$.

1a. What is the order of A?
The matrix has 3 rows and 2 columns, so it is of order 3×2.

1b. Identify a_{12} and a_{31}.

The element a_{12} is in the first row and second column: $a_{12} = -2$.

The element a_{31} is in the third row and first column: $a_{31} = 1$.

✎ *Pencil Problem #1* ✐

1. Let $A = \begin{bmatrix} 1 & -5 & \pi & e \\ 0 & 7 & -6 & -\pi \\ -2 & \frac{1}{2} & 11 & -\frac{1}{5} \end{bmatrix}$.

1a. What is the order of A?

1b. Identify a_{32} and a_{23}.

Objective #2: Understand what is meant by equal matrices.

✔ *Solved Problem #2*

2. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x & y+1 \\ z & 6 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}$$

These matrices are of the same order, so they are equal if and only if corresponding elements are equal.

$x = 1$
$y + 1 = 5$, so $y = 4$
$z = 3$

✎ *Pencil Problem #2* ✐

2. Find values for the variables so that the matrices are equal.

$$\begin{bmatrix} x & 2y \\ z & 9 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 3 & 9 \end{bmatrix}$$

Objective #3: Add and subtract matrices.

✔ *Solved Problem #3*

3. Perform the indicated matrix operations.

3a. $\begin{bmatrix} -4 & 3 \\ 7 & -6 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 2 & -4 \end{bmatrix}$

Add corresponding elements.

$$\begin{bmatrix} -4+6 & 3+(-3) \\ 7+2 & -6+(-4) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 9 & -10 \end{bmatrix}$$

3b. $\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$

Subtract corresponding elements.

$$\begin{bmatrix} 5-(-4) & 4-8 \\ -3-6 & 7-0 \\ 0-(-5) & 1-3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{bmatrix}$$

✎ *Pencil Problem #3* ✎

3. Perform the indicated matrix operations.

3a. $\begin{bmatrix} 1 & 3 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}$

3b. $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$

Objective #4: Perform scalar multiplication.

✔ *Solved Problem #4*

4. If $A = \begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$, find each of the following.

4a. $-6B$

$$\begin{aligned} -6B &= -6\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -6(-1) & -6(-2) \\ -6(8) & -6(5) \end{bmatrix} \\ &= \begin{bmatrix} 6 & 12 \\ -48 & -30 \end{bmatrix} \end{aligned}$$

✎ *Pencil Problem #4* ✎

4. If $A = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 \\ 3 \\ -1 \end{bmatrix}$, find each of the following.

4a. $-4A$

4b. $3A + 2B$

$$3A+2B = 3\begin{bmatrix} -4 & 1 \\ 3 & 0 \end{bmatrix} + 2\begin{bmatrix} -1 & -2 \\ 8 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 3(-4) & 3(1) \\ 3(3) & 3(0) \end{bmatrix} + \begin{bmatrix} 2(-1) & 2(-2) \\ 2(8) & 2(5) \end{bmatrix}$$
$$= \begin{bmatrix} -12 & 3 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 16 & 10 \end{bmatrix}$$
$$= \begin{bmatrix} -12+(-2) & 3+(-4) \\ 9+16 & 0+10 \end{bmatrix}$$
$$= \begin{bmatrix} -14 & -1 \\ 25 & 10 \end{bmatrix}$$

4b. $3A + 2B$

Objective #5: Solve matrix equations.

✔ *Solved Problem #5*

5. Solve for X in the matrix equation $3X + A = B$ where $A = \begin{bmatrix} 2 & -8 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & 1 \\ -9 & 17 \end{bmatrix}$.

Begin by solving the matrix equation for X.

$$3X + A = B$$
$$3X = B - A$$
$$X = \frac{1}{3}(B - A)$$

Now use matrices A and B to find X.

$$X = \frac{1}{3}\left(\begin{bmatrix} -10 & 1 \\ -9 & 17 \end{bmatrix} - \begin{bmatrix} 2 & -8 \\ 0 & 4 \end{bmatrix}\right)$$
$$= \frac{1}{3}\begin{bmatrix} -12 & 9 \\ -9 & 13 \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 3 \\ -3 & \frac{13}{3} \end{bmatrix}$$

✎ *Pencil Problem #5* ✎

5. Solve for X in the matrix equation $2X + A = B$ where $A = \begin{bmatrix} -3 & -7 \\ 2 & -9 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -1 \\ 0 & 0 \\ 3 & -4 \end{bmatrix}$.

Objective #6: Multiply matrices.

Solved Problem #6

6a. Find AB, given $A=\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}4 & 6\\ 1 & 0\end{bmatrix}$.

$$AB=\begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}\begin{bmatrix}4 & 6\\ 1 & 0\end{bmatrix}$$
$$=\begin{bmatrix}1(4)+3(1) & 1(6)+3(0)\\ 2(4)+5(1) & 2(6)+5(0)\end{bmatrix}$$
$$=\begin{bmatrix}7 & 6\\ 13 & 12\end{bmatrix}$$

6b. Find the product, if possible.
$$\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 3 & -1 & 6\\ 0 & 5 & 4 & 1\end{bmatrix}$$

The number of columns in the first matrix equals the number of rows in the second matrix, so it is possible to find the product.

$$\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 3 & -1 & 6\\ 0 & 5 & 4 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1(2)+3(0) & 1(3)+3(5) & 1(-1)+3(4) & 1(6)+3(1)\\ 0(2)+2(0) & 0(3)+2(5) & 0(-1)+2(4) & 0(6)+2(1)\end{bmatrix}$$
$$=\begin{bmatrix}2 & 18 & 11 & 9\\ 0 & 10 & 8 & 2\end{bmatrix}$$

6c. Find the product, if possible.
$$\begin{bmatrix}2 & 3 & -1 & 6\\ 0 & 5 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 0 & 2\end{bmatrix}$$

The number of columns in the first matrix does not equal the number of rows in the second matrix. The product of the matrices is undefined.

Pencil Problem #6

6a. Find AB, given $A=\begin{bmatrix}1 & 3\\ 5 & 3\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2\\ -1 & 6\end{bmatrix}$.

6b. Find the product, if possible.
$$\begin{bmatrix}4 & 2\\ 6 & 1\\ 3 & 5\end{bmatrix}\begin{bmatrix}2 & 3 & 4\\ -1 & -2 & 0\end{bmatrix}$$

6c. Find the product, if possible.
$$\begin{bmatrix}2 & 3 & 4\\ -1 & -2 & 0\end{bmatrix}\begin{bmatrix}4 & 2\\ 6 & 1\\ 3 & 5\end{bmatrix}$$

Objective #7: Model applied situations with matrix operations.

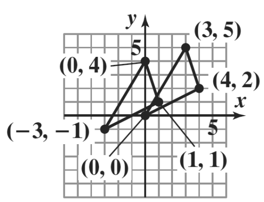
✔ *Solved Problem #7*

7. The triangle with vertices (0, 0), (3, 5), and (4, 2) in a rectangular coordinate system can be represented by the matrix $\begin{bmatrix} 0 & 3 & 4 \\ 0 & 5 & 2 \end{bmatrix}$. Use matrix operations to move the triangle 3 units to the left and 1 unit down. Graph the original triangle and the transformed triangle in the same rectangular coordinate system.

We subtract 3 from each x-coordinate and subtract 1 from each y-coordinate.

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 5 & 2 \end{bmatrix} + \begin{bmatrix} -3 & -3 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 0 & 1 \\ -1 & 4 & 1 \end{bmatrix}$$

The vertices of the translated triangle are (–3, –4), (0, 4), and (1, 1).



Pencil Problem #7

7. An L-shaped figure has vertices at (0, 0), (3, 0), (3, 1), (1, 1), (1, 5), and (0, 5) in a rectangular coordinate system and can be represented by the matrix $\begin{bmatrix} 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 5 & 5 \end{bmatrix}$. Use matrix operations to move the figure 2 units to the left and 3 units down.
Graph the original figure and the transformed figure in the same rectangular coordinate system.

<u>Answers</u> <u>for Pencil Problems</u> *<u>(Textbook Exercise references in parentheses)</u>*:

1a. 3×4 *(8.3 #3a)* **1b.** $a_{32} = \frac{1}{2}$; $a_{23} = -6$ *(8.3 #3b)*

2. $x = 4$, $y = 6$, $z = 3$ *(8.3 #7)*

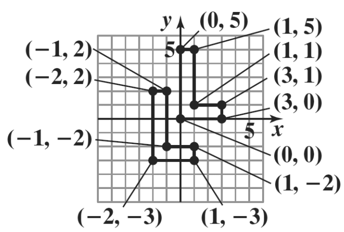
3a. $\begin{bmatrix} 3 & 2 \\ 6 & 2 \\ 5 & 7 \end{bmatrix}$ *(8.3 #11a)* **3b.** $\begin{bmatrix} -1 & -8 \\ 3 & -5 \end{bmatrix}$ *(8.3 #9b)*

4a. $\begin{bmatrix} -8 \\ 16 \\ -4 \end{bmatrix}$ *(8.3 #13c)* **4b.** $\begin{bmatrix} -4 \\ -6 \\ 1 \end{bmatrix}$ *(8.3 #13d)*

5. $X = \begin{bmatrix} -1 & 3 \\ -1 & \frac{9}{2} \\ -1 & -2 \end{bmatrix}$ *(8.3 #19)*

6a. $\begin{bmatrix} 0 & 16 \\ 12 & 8 \end{bmatrix}$ *(8.3 #27a)* **6b.** $\begin{bmatrix} 6 & 8 & 16 \\ 11 & 16 & 24 \\ 1 & -1 & 12 \end{bmatrix}$ *(8.3 #33a)* **6c.** $\begin{bmatrix} 38 & 27 \\ -16 & -4 \end{bmatrix}$ *(8.3 #33b)*

7. $\begin{bmatrix} -2 & 1 & 1 & -1 & -1 & -2 \\ -3 & -3 & -2 & -2 & 2 & 2 \end{bmatrix}$;



(8.3 #53)

Section 8.4
Multiplicative Inverses of Matrices and Matrix Equations

THINK YOU'RE TOO OLD FOR SENDING SECRET MESSAGES?

Did you know that a secure electronic message is usually encrypted in such a way that only the receiver will be able to decrypt it to read it? The technology used is a bit more sophisticated than the cereal-box ciphers that we used to send secret messages as kids.

Matrix multiplication can be used to encrypt a message. To decrypt the message, we need to know the multiplicative inverse of the encryption matrix. Without the original matrix or its inverse, it is extremely difficult to "break the code."

Objective #1: Find the multiplicative inverse of a square matrix.

✔ *Solved Problem #1*

1a. Find the multiplicative inverse of $A=\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$.

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$, so $a = 3$, $b = -2$, $c = -1$, and $d = 1$.

$ad-bc=3(1)-(-2)(-1)=1\neq 0,$ so the matrix has an inverse.

Using the quick method,

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$=\frac{1}{3(1)-(-2)(-1)}\begin{bmatrix} 1 & -(-2) \\ -(-1) & 3 \end{bmatrix}$$

$$=\frac{1}{1}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

You can verify the result by showing that $AA^{-1}=I_2$ and $A^{-1}A=I_2$, where $I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix.

✎ *Pencil Problem #1* ✐

1a. Find the multiplicative inverse of $A=\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$.

1b. Find the multiplicative inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Form the augmented matrix $[A|I_3]$ and perform row operations to obtain a matrix of the form $[I_3|B]$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[-\frac{5}{2}R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 3 & -2 & -5 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right]$$

The inverse matrix is

$$A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}.$$

You can verify the result by showing that $AA^{-1} = I_3$ and $A^{-1}A = I_3$.

1b. Find the multiplicative inverse of

$A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

Objective #2: Use inverses to solve matrix equations.

✔ *Solved Problem #2*

2. Solve the system by using A^{-1}, the inverse of the coefficient matrix, where $A^{-1}=\begin{bmatrix}3&-2&-4\\3&-2&-5\\-1&1&2\end{bmatrix}$.

$$\begin{cases}x+2z=6\\-x+2y+3z=-5\\x-y=6\end{cases}$$

The system can be written as

$$\begin{bmatrix}1&0&2\\-1&2&3\\1&-1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-5\\6\end{bmatrix},$$

which is of the form $AX=B$. The solution is $X=A^{-1}B$.

$$X=A^{-1}B=\begin{bmatrix}3&-2&-4\\3&-2&-5\\-1&1&2\end{bmatrix}\begin{bmatrix}6\\-5\\6\end{bmatrix}$$

$$=\begin{bmatrix}3(6)-2(-5)-4(6)\\3(6)-2(-5)-5(6)\\-1(6)+1(-5)+2(6)\end{bmatrix}$$

$$=\begin{bmatrix}4\\-2\\1\end{bmatrix}$$

So, $x=4$, $y=-2$, and $z=1$.
The solution set is $\{(4,-2,1)\}$.

✎ *Pencil Problem #2* ✎

2. Solve the system by using A^{-1}, the inverse of the coefficient matrix, where

$$A^{-1}=\begin{bmatrix}3&3&-1\\-2&-2&1\\-4&-5&2\end{bmatrix}.$$

$$\begin{cases}x-y+z=8\\2y-z=-7\\2x+3y=1\end{cases}$$

Objective #3: Encode and decode messages.

✔ *Solved Problem #3*

3a. Use the coding matrix $\begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}$ to encode the word BASE.

The numerical equivalent of the word BASE is 2, 1, 19, 5.

The matrix for the word BASE is $\begin{bmatrix} 2 & 19 \\ 1 & 5 \end{bmatrix}$.

Multiply using the encoding matrix on the left.

$$\begin{bmatrix} -2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 19 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -2(2)-3(1) & -2(19)-3(5) \\ 3(2)+4(1) & 3(19)+4(5) \end{bmatrix}$$
$$= \begin{bmatrix} -7 & -53 \\ 10 & 77 \end{bmatrix}$$

The encoded message is –7, 10, –53, 77.

3b. Decode the word encoded in Solved Problem 3a.
Find the inverse of the coding matrix.

$$A^{-1} = \frac{1}{-2(4)-(-3)(3)}\begin{bmatrix} 4 & -(-3) \\ -3 & -2 \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

Multiply A^{-1} and the coded matrix from Solved Problem 3a.

$$\begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} -7 & -53 \\ 10 & 77 \end{bmatrix}$$
$$= \begin{bmatrix} 4(-7)+3(10) & 4(-53)+3(77) \\ -3(-7)-2(10) & -3(-53)-2(77) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 19 \\ 1 & 5 \end{bmatrix}$$

The decoded message is 2, 1, 19, 5, or BASE.

✎ *Pencil Problem #3* ✐

3a. Use the coding matrix $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ to encode the word HELP.

3b. Decode the word encoded in Pencil Problem 3a.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{3}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$ *(8.4 #13)* **1b.** $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ *(8.4 #21)*

2. $\{(2, -1, 5)\}$ *(8.4 #39)*

3a. 27, –19, 32, –20 *(8.4 #51)* **3b.** 8, 5, 12, 16 or HELP *(8.4 #51)*

Section 8.5
Determinants and Cramer's Rule

Look......Closer!!!

Do you see the difference between these two mathematical expressions?

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

If you look carefully, you will notice that the expression on the left is surrounded by brackets, [ ], and is therefore a **matrix**. The expression on the right is surrounded by bars, | |, and represents a **determinant**.

But be careful! This section will discuss *both* determinants *and* matrices.

Objective #1: Evaluate a second-order determinant.

✔ *Solved Problem #1*

1. Evaluate the determinant of the matrix: $\begin{bmatrix} 10 & 9 \\ 6 & 5 \end{bmatrix}$

The determinant of the matrix $\begin{bmatrix} 10 & 9 \\ 6 & 5 \end{bmatrix}$ is $\begin{vmatrix} 10 & 9 \\ 6 & 5 \end{vmatrix}$.

$$\begin{vmatrix} 10 & 9 \\ 6 & 5 \end{vmatrix} = 10(5) - 6(9) = 50 - 54 = -4$$

✎ *Pencil Problem #1* ✎

1. Evaluate the determinant: $\begin{vmatrix} -4 & 1 \\ 5 & 6 \end{vmatrix}$

Objective #2: Solve a system of linear equations in two variables using Cramer's rule.

✔ *Solved Problem #2*

2. Use Cramer's rule to solve the system:

$$\begin{cases} 5x + 4y = 12 \\ 3x - 6y = 24 \end{cases}$$

$$D = \begin{vmatrix} 5 & 4 \\ 3 & -6 \end{vmatrix} = 5(-6) - 3(4) = -30 - 12 = -42$$

$$D_x = \begin{vmatrix} 12 & 4 \\ 24 & -6 \end{vmatrix} = 12(-6) - 24(4) = -72 - 96 = -168$$

$$D_y = \begin{vmatrix} 5 & 12 \\ 3 & 24 \end{vmatrix} = 5(24) - 3(12) = 120 - 36 = 84$$

$$x = \frac{D_x}{D} = \frac{-168}{-42} = 4 \quad y = \frac{D_y}{D} = \frac{84}{-42} = -2$$

The solution set is $\{(4, -2)\}$.

✎ *Pencil Problem #2* ✎

2. Use Cramer's rule to solve the system:

$$\begin{cases} 12x + 3y = 15 \\ 2x - 3y = 13 \end{cases}$$

Objective #3: Evaluate a third-order determinant.

✔ *Solved Problem #3*

3. Evaluate the determinant of the matrix:

$$\begin{bmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 7 \\ -5 & 6 & 0 \\ -4 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 0 \\ 3 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 1 & 7 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 7 \\ 6 & 0 \end{vmatrix}$$

$$= 2\left(6(1) - 3(0)\right) + 5\left(1(1) - 3(7)\right) - 4\left(1(0) - 6(7)\right)$$

$$= 2(6) + 5(-20) - 4(-42)$$

$$= 12 - 100 + 168$$

$$= 80$$

✎ *Pencil Problem #3* ✐

3. Evaluate the determinant:

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & -5 \\ 2 & 5 & -1 \end{vmatrix}$$

Objective #4: Solve a system of linear equations in three variables using Cramer's rule.

✔ *Solved Problem #4*

4. Use Cramer's rule to solve the system:

$$\begin{cases} 3x - 2y + z = 16 \\ 2x + 3y - z = -9 \\ x + 4y + 3z = 2 \end{cases}$$

First, find D, D_x, D_y, and D_z.

$$D = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 3 & -1 \\ 1 & 4 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 58$$

$$D_x = \begin{vmatrix} 16 & -2 & 1 \\ -9 & 3 & -1 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= 16\begin{vmatrix} 3 & -1 \\ 4 & 3 \end{vmatrix} - (-9)\begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 116$$

✎ *Pencil Problem #4* ✐

4. Use Cramer's rule to solve the system:

$$\begin{cases} x + y + z = 0 \\ 2x - y + z = -1 \\ -x + 3y - z = -8 \end{cases}$$

$$D_y = \begin{vmatrix} 3 & 16 & 1 \\ 2 & -9 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$
$$= 3\begin{vmatrix} -9 & -1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 16 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} 16 & 1 \\ -9 & -1 \end{vmatrix}$$
$$= -174$$

$$D_z = \begin{vmatrix} 3 & -2 & 16 \\ 2 & 3 & -9 \\ 1 & 4 & 2 \end{vmatrix}$$
$$= 3\begin{vmatrix} 3 & -9 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} -2 & 16 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} -2 & 16 \\ 3 & -9 \end{vmatrix}$$
$$= 232$$

Next, use D, D_x, D_y, and D_z to find x, y, and z.

$D = 58$, $D_x = 116$, $D_y = -174$, and $D_z = 232$.

$$x = \frac{D_x}{D} = \frac{116}{58} = 2$$

$$y = \frac{D_y}{D} = \frac{-174}{58} = -3$$

$$z = \frac{D_z}{D} = \frac{232}{58} = 4$$

The solution set is $\{(2, -3, 4)\}$.

Objective #5: Evaluate higher-order determinants.

✔ *Solved Problem #5*

5. Evaluate the determinant: $\begin{vmatrix} 0 & 4 & 0 & -3 \\ -1 & 1 & 5 & 2 \\ 1 & -2 & 0 & 6 \\ 3 & 0 & 0 & 1 \end{vmatrix}$.

With three 0s in the third column, we expand along the third column.

$$\begin{vmatrix} 0 & 4 & 0 & -3 \\ -1 & 1 & 5 & 2 \\ 1 & -2 & 0 & 6 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1)^{2+3}(5)\begin{vmatrix} 0 & 4 & -3 \\ 1 & -2 & 6 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= -5\left((-1)^{2+1}(1)\begin{vmatrix} 4 & -3 \\ 0 & 1 \end{vmatrix} + (-1)^{3+1}(3)\begin{vmatrix} 4 & -3 \\ -2 & 6 \end{vmatrix}\right)$$

$$= -5\big(-1(4(1)-(-3)(0)) + 3(4(6)-(-3)(-2))\big)$$

$$= -5(-1(4-0)+3(24-6))$$

$$= -5(-4+54)$$

$$= -5(50)$$

$$= -250$$

✎ *Pencil Problem #5* ✐

5. Evaluate the determinant: $\begin{vmatrix} 4 & 2 & 8 & -7 \\ -2 & 0 & 4 & 1 \\ 5 & 0 & 0 & 5 \\ 4 & 0 & 0 & -1 \end{vmatrix}$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. -29 *(8.5 #3)*

2. $\{(2,-3)\}$ *(8.5 #13)*

3. 72 *(8.5 #23)*

4. $\{(-5,-2,7)\}$ *(8.5 #29)*

5. -200 *(8.5 #37)*