Section 6.1
The Law of Sines

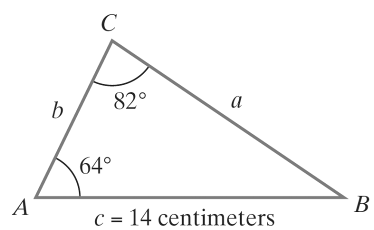
I SEE A FIRE, BUT WHERE IS IT?

We have solved many applied problems using the properties of right triangles and the relationships among the sides and angles. However, not every situation involving a triangle involves a right triangle. In this section and the next, we look at two laws that allow us to solve problems involving oblique triangles. In this section's Exercise Set, you will see how we can locate a fire, measure the height of a tree on a hillside, or find the distance a shot put is tossed using the Law of Sines.

Objective #1: Use the Law of Sines to solve oblique triangles.

✔ *Solved Problem #1*

1a. Solve the triangle shown in the figure. Round lengths of sides to the nearest tenth.



We know two angles and the side adjacent to one of the angles. This is an SAA triangle and the Law of Sines applies.

First, find the third angle.

$$\begin{aligned} A+B+C &= 180^\circ \\ 64^\circ+B+82^\circ &= 180^\circ \\ B+146^\circ &= 180^\circ \\ B &= 34^\circ \end{aligned}$$

We know the length of one side, $c = 14$ centimeters, and the measure of the angle opposite this side, $C = 82°$. We use the ratio $\frac{14}{\sin 82^\circ}$ and the Law of Sines to find the lengths of the other two sides.

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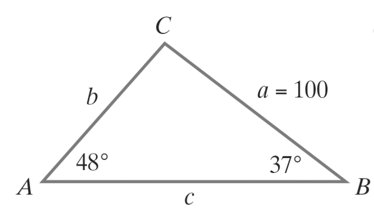
✎ *Pencil Problem #1* ✎

1a. Solve the triangle shown in the figure. Round lengths of sides to the nearest tenth.



$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 64°} = \frac{14}{\sin 82°} \qquad \frac{b}{\sin 34°} = \frac{14}{\sin 82°}$$
$$a = \frac{14 \sin 64°}{\sin 82°} \qquad b = \frac{14 \sin 34°}{\sin 82°}$$
$$a \approx 12.7 \text{ cm} \qquad b \approx 7.9 \text{ cm}$$

Thus, $B = 34°$, $a \approx 12.7$ cm, and $b \approx 7.9$ cm.

1b. Solve triangle ABC if $A = 40°$, $C = 22.5°$, and $b = 12$. Round lengths of sides to the nearest tenth.

You may want to draw a triangle and label it with the given information. We know two angles and the side between the angles. This is an ASA triangle and the Law of Sines applies.

First, find the third angle.

$$A + B + C = 180°$$
$$40° + B + 22.5° = 180°$$
$$B + 62.5° = 180°$$
$$B = 117.5°$$

We know the length of one side, $b = 12$, and the measure of the angle opposite this side, $B = 117.5°$. We use the ratio $\frac{12}{\sin 117.5°}$ and the Law of Sines to find the lengths of the other two sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{c}{\sin C} = \frac{b}{\sin B}$$
$$\frac{a}{\sin 40°} = \frac{12}{\sin 117.5°} \qquad \frac{c}{\sin 22.5°} = \frac{12}{\sin 117.5°}$$
$$a = \frac{12 \sin 40°}{\sin 117.5°} \qquad c = \frac{12 \sin 22.5°}{\sin 117.5°}$$
$$a \approx 8.7 \qquad c \approx 5.2$$

Thus, $B = 117.5°$, $a \approx 8.7$, and $c \approx 5.2$ cm.

1b. Solve triangle ABC if $A = 65°$, $B = 65°$, and $c = 6$. Round lengths of sides to the nearest tenth.

Objective #2: Use the Law of Sines to solve, if possible, the triangle or triangles in the ambiguous case.

✔ Solved Problem #2

2a. Solve triangle ABC if $A = 57°$, $a = 33$, and $b = 26$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

We know two sides and the angle opposite one of the sides. This is an SSA case, where we first need to determine whether no triangle, exactly one triangle, or two triangles that satisfy the conditions exist.

We use the Law of Sines to find $\sin B$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{33}{\sin 57°} = \frac{26}{\sin B}$$
$$33 \sin B = 26 \sin 57°$$
$$\sin B = \frac{26 \sin 57°}{33} \approx 0.6608$$

There are two angles between 0° and 180° that satisfy $\sin B \approx 0.6608$:

$B_1 \approx \sin^{-1} 0.6608 \approx 41°$ and
$B_2 \approx 180° - 41° = 139°$.

Note that the angle sum of the given angle, $A = 57°$, and $B_2 \approx 139°$ is 196°, which exceeds 180°, so no triangle is possible with these two angles. However, this problem does not occur for angle B_1, so we have exactly one triangle.

We let $B = B_1 \approx 41°$. Now we find C and c.

$$A + B + C = 180°$$
$$57° + 41° + C \approx 180°$$
$$C + 98° \approx 180°$$
$$C \approx 82°$$

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✎ Pencil Problem #2 ✎

2a. Solve triangle ABC if $A = 40°$, $a = 20$, and $b = 15$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

Use the Law of Sines to find c. We try to use exact values where possible, so we choose to use A in our second ratio instead of B, since B is rounded. We have no choice but to use the value for C that we calculated in the previous step.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 82^\circ} = \frac{33}{\sin 57^\circ}$$
$$c = \frac{33 \sin 82^\circ}{\sin 57^\circ} \approx 39.0$$

Thus, there is one triangle with $B \approx 41^\circ$, $C \approx 82^\circ$, and $c \approx 39.0$.

2b. Solve triangle ABC if $A = 50^\circ$, $a = 10$, and $b = 20$.

We know two sides and the angle opposite one of the sides. This is an SSA case, where we first need to determine whether no triangle, exactly one triangle, or two triangles that satisfy the conditions exist.

We use the Law of Sines to find $\sin B$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{10}{\sin 50^\circ} = \frac{20}{\sin B}$$
$$10 \sin B = 20 \sin 50^\circ$$
$$\sin B = \frac{20 \sin 50^\circ}{10} \approx 1.5321$$

Because the value of sine cannot exceed 1, there is no value of B for which $\sin B \approx 1.5321$. There is no triangle with the given measurements.

2b. Solve triangle ABC if $A = 30^\circ$, $a = 10$, and $b = 40$.

2c. Solve triangle ABC if $A = 35°$, $a = 12$, and $b = 16$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

We know two sides and the angle opposite one of the sides. This is an SSA case, where we first need to determine whether no triangle, exactly one triangle, or two triangles that satisfy the conditions exist.

We use the Law of Sines to find $\sin B$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{12}{\sin 35°} = \frac{16}{\sin B}$$
$$12\sin B = 16\sin 35°$$
$$\sin B = \frac{16\sin 35°}{12} \approx 0.7648$$

There are two angles between 0° and 180° that satisfy $\sin B \approx 0.7648$:

$B_1 \approx \sin^{-1} 0.7648 \approx 50°$ and
$B_2 \approx 180° - 50° = 130°$.

Note that the given angle, $A = 35°$, can be added to either of these angles without exceeding 180°, so we have two triangles. Find C_1 and C_2.

$$\begin{aligned} A + B_1 + C_1 &= 180° & A + B_2 + C_2 &= 180° \\ 35° + 50° + C_1 &\approx 180° & 35° + 130° + C_2 &\approx 180° \\ C_1 + 85° &\approx 180° & C_2 + 165° &\approx 180° \\ C_1 &\approx 95° & C_2 &\approx 15° \end{aligned}$$

Now find c_1 and c_2. As in Solved Problem #2a, we use exact values where possible.

$$\begin{aligned} \frac{c_1}{\sin C_1} &= \frac{a}{\sin A} & \frac{c_2}{\sin C_2} &= \frac{a}{\sin A} \\ \frac{c_1}{\sin 95°} &= \frac{12}{\sin 35°} & \frac{c_2}{\sin 15°} &= \frac{12}{\sin 35°} \\ c_1 &= \frac{12\sin 95°}{\sin 35°} & c_2 &= \frac{12\sin 15°}{\sin 35°} \\ c_1 &\approx 20.8 & c_2 &\approx 5.4 \end{aligned}$$

In summary, there are two possible triangles with the given measurements. One has $B_1 \approx 50°$, $C_1 \approx 95°$, and $c_1 \approx 20.8$. The other has $B_2 \approx 130°$, $C_2 \approx 15°$, and $c_2 \approx 5.4$.

2c. Solve triangle ABC if $A = 60°$, $a = 16$, and $b = 18$. Round lengths of sides to the nearest tenth and angle measures to the nearest degree.

Objective #3: Find the area of an oblique triangle using the sine function.

✔ Solved Problem #3

3. Find the area of a triangle having two sides of lengths 8 meters and 12 meters and an included angle of 135°. Round to the nearest square meter.

$$\text{Area } = \frac{1}{2}(8)(12)\sin 135° \approx 34 \text{ m}^2$$

✎ Pencil Problem #3 ✎

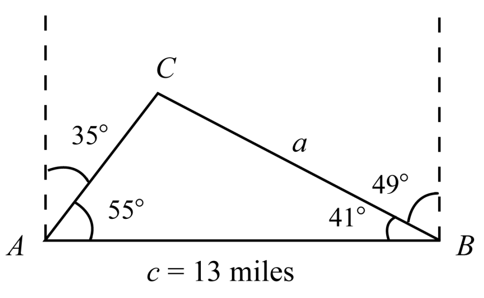
3. Find the area of a triangle having two sides of lengths 20 feet and 40 feet and an included angle of 48°. Round to the nearest square foot.

Objective #4: Solve applied problems using the Law of Sines.

✔ Solved Problem #4

4. Two fire-lookout stations are 13 miles apart, with station B directly east of station A. Both stations spot a fire. The bearing of the fire from station A is N35°E and the bearing of the fire from station B is N49°W. How far, to the nearest tenth of a mile, is the fire from station B?

Refer to the figure. Note that the angle measurements given in the problem are not interior angles of the triangle shown. To find the respective interior angles, we find the complements of the given angles: $A = 90° - 35° = 55°$ and $B = 90° - 49° = 41°$.



✎ Pencil Problem #4 ✎

4. Two fire-lookout stations are 10 miles apart, with station B directly east of station A. Both stations spot a fire. The bearing of the fire from station A is N25°E and the bearing of the fire from station B is N56°W. How far, to the nearest tenth of a mile, is the fire from station B?

The fire is at C. We need to find the distance of the fire from B. This distance is labeled a in the figure, since it is opposite angle A. The only known side is c, so we need to find angle C before we can use the Law of Sines.

$C = 180° - 55° - 41° = 84°$

Now we use the Law of Sines to find a.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 55°} = \frac{13}{\sin 84°}$$
$$a = \frac{13 \sin 55°}{\sin 84°}$$
$$a \approx 10.7$$

The fire is approximately 10.7 miles from station B.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

Note: It is possible to have slightly different answers due to rounding in intermediate steps.

1a. $C = 95°$, $b \approx 81.0$, $c \approx 134.1$ *(6.1 #5)* **1b.** $C = 50°$, $a \approx 7.1$, $b \approx 7.1$ *(6.1 #15)*

2a. one triangle: $B \approx 29°$, $C \approx 111°$, $c \approx 29.0$ *(6.1 #17)* **2b.** no triangle *(6.1 #23)*
2c. two triangles: $B_1 \approx 77°$, $C_1 \approx 43°$, $c_1 \approx 12.6$ and $B_2 \approx 103°$, $C_2 \approx 17°$, $c_2 \approx 5.4$ *(6.1 #25)*

3. 297 square feet *(6.1 #33)*

4. approximately 9.2 miles *(6.1 #47)*

Section 6.2
The Law of Cosines

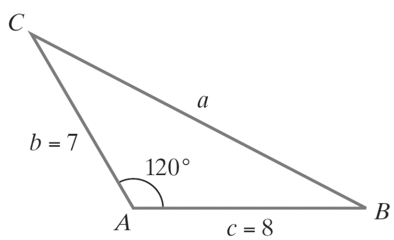
Jurassic Math

Fossil footprints allow scientists to study the movement of dinosaurs. In this section, you will learn the Law of Cosines which will help you determine whether a dinosaur was an efficient walker from its footprints.
Like the Law of Sines, the Law of Cosines is used to solve oblique triangles. The information given in a problem will determine which one you use. Be sure to pay attention not only to how to use the formulas but also to which formula to use.

Objective #1: Use the Law of Cosines to solve oblique triangles.

✔ *Solved Problem #1*

1a. Solve the triangle shown in the figure. Round lengths of sides to the nearest tenth and angles to the nearest degree.



We know two sides and the included angle. This is an SAS triangle. In this case, we use the Law of Cosines to find the missing side, the side opposite the known angle, first. Since this is side *a*, we use the Law of Cosines in the form
$a^2 = b^2 + c^2 - 2ab\cos A$, with $b = 7$, $c = 8$, and $A = 120°$. Use a calculator in degree mode to simplify the right side.

$$a^2 = 7^2 + 8^2 - 2(7)(8)\cos 120° = 169$$
$$a = \sqrt{169} = 13$$

Now we use the Law of Sines to find the angle opposite the shorter of the two given sides, $b = 7$ and $c = 8$. Since *b* is shorter, we find angle *B*.

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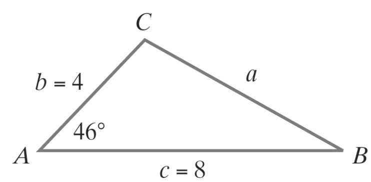
✎ *Pencil Problem #1* ✎

1a. Solve the triangle shown in the figure. Round lengths of sides to the nearest tenth and angles to the nearest degree.



$$\frac{b}{\sin B}=\frac{a}{\sin A}$$
$$\frac{7}{\sin B}=\frac{13}{\sin 120°}$$
$$13\sin B = 7\sin 120°$$
$$\sin B=\frac{7\sin 120°}{13}\approx 0.4663$$
$$B\approx \sin^{-1} 0.4663 \approx 28°$$

(Since the angle opposite the shorter side must be acute, we do not need to consider a value for *B* in quadrant II.)

Now we find angle *C*.

$$C \approx 180° - 120° - 28° = 32°$$

In summary, $a = 13$, $B \approx 28°$, and $C \approx 32°$.

1b. Solve triangle *ABC* if $a = 8$, $b = 10$, and $c = 5$. Round angle measures to the nearest degree.

In this case, we know the lengths of all three sides. This is an SSS situation. We use the Law of Cosines first to find the angle opposite the longest side. (If the triangle has an obtuse angle, we want to find it first.) The longest side is $b = 10$, so we find angle *B* first. We use the form of the Law of Cosines that contains *B*. Begin by solving the formula for $\cos B$.

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$2ac\cos B = a^2 + c^2 - b^2$$
$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

Now substitute the given values, $a = 8$, $b = 10$, and $c = 5$, on the right side and simplify.

$$\cos B = \frac{8^2+5^2-10^2}{2(8)(5)} = -\frac{11}{80}$$

Now solve for *B*.

$$B=\cos^{-1}\left(-\frac{11}{80}\right)\approx 98°$$

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1b. Solve triangle *ABC* if $a = 5$, $b = 7$, and $c = 10$. Round angle measures to the nearest degree.

The two remaining angles are both acute. We will use the Law of Sines to find one of them. We will find angle A.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{8}{\sin A} = \frac{10}{\sin 98°}$$
$$10\sin A = 8\sin 98°$$
$$\sin A = \frac{8\sin 98°}{10} \approx 0.7922$$
$$B \approx \sin^{-1} 0.7922 \approx 52°$$

Finally, we find C.

$$C \approx 180° - 52° - 98° = 30°$$

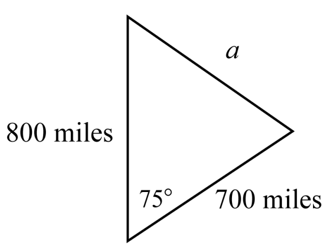
In summary, $A \approx 52°$, $B \approx 98°$, and $C \approx 30°$.

Objective #2: Solve applied problems using the Law of Cosines.

✔ *Solved Problem #2*

2. Two airplanes leave an airport at the same time on different runways. One flies directly north at 400 miles per hour. The other airplane flies on a bearing of N75°E at 350 miles per hour. How far apart will the airplanes be after two hours?

Refer to the figure. The first airplane has flown 800 miles in two hours, and the second has flown 700 miles. The angle between their paths is 75°. We are asked to find the distance between the planes, which is the length of the side opposite the 75° angle.



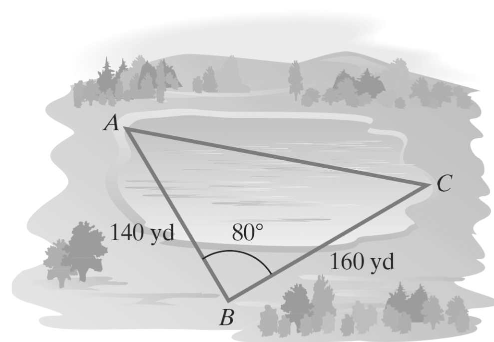
Since we know two sides and the included angle (SAS), we use the Law of Cosines. We substitute b = 800, c = 700, and A = 75° into the right side of the formula and simplify with a calculator.

$$a^2 = b^2 + c^2 - 2ab\cos A$$
$$a^2 = 800^2 + 700^2 - 2(800)(700)\cos 75° \approx 840{,}123$$
$$a \approx \sqrt{840{,}123} \approx 917$$

The planes are approximately 917 miles apart after two hours.

✎ *Pencil Problem #2* ✎

2. Find the distance across the lake from A to C, to the nearest yard, using the measurements shown in the figure.



Objective #3: Use Heron's formula to find the area of a triangle.

✔ *Solved Problem #3*

3. Find the area of a triangle with a = 6 meters, b = 16 meters, and c = 18 meters. Round to the nearest meter.

To use Heron's formula, we first find s, one-half the perimeter.

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(6+16+18) = 20$$

Now we use Heron's formula to find the area.

$$\begin{aligned} \text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{20(20-6)(20-16)(20-18)} \\ &= \sqrt{2240} \approx 47 \text{ m}^2 \end{aligned}$$

✎ *Pencil Problem #3* ✐

3. Find the area of a triangle with a = 11 yards, b = 9 yards, and c = 7 yards. Round to the nearest square yard.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

Note: It is possible to have slightly different answers due to rounding in intermediate steps.

1a. $a \approx 6.0$, $B \approx 29°$, $C \approx 105°$ *(6.2 #1)* **1b.** $C \approx 112°$, $A \approx 28°$, $B \approx 40°$ *(6.2 #17)*

2. approximately 193 yards *(6.2 #41)*

3. 31 square yards *(6.2 #29)*

Section 6.3
Polar Coordinates

Getting a New Perspective

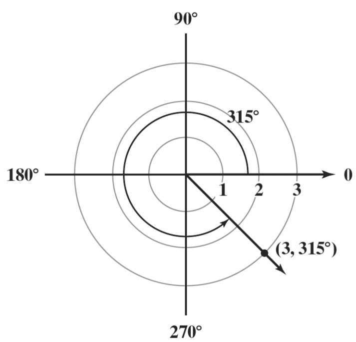
We have worked extensively in the Cartesian coordinate system, plotting points, graphing equations, and using the properties of the Cartesian plane to investigate functions and solve problems. In this section, we introduce a different coordinate system that will give you a new perspective of the plane. In this system, points do not have unique ordered pairs associated with them and some complicated-looking graphs have very simple equations.

Objective #1: Plot points in the polar coordinate system.

✔ *Solved Problem #1*

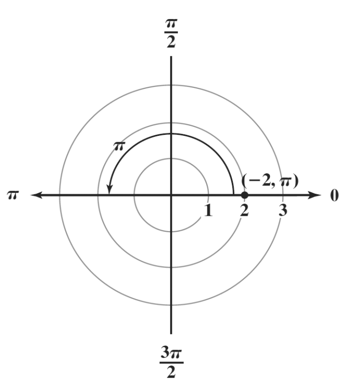
1a. Plot the point with polar coordinates (3, 315°).

Draw an angle measuring 315° in standard position. Then plot a point 3 units from the origin (pole) along the terminal side of the angle.



1b. Plot the point with polar coordinates (−2, π).

Draw an angle measuring π radians in standard position. Then plot a point 2 units from the origin (pole) along a line in the opposite direction from the terminal side of the angle. Since the terminal side of the angle points to the left, this means that we move 2 units to the right.



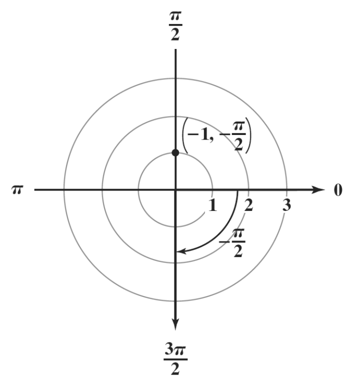
✎ *Pencil Problem #1* ✎

1a. Plot the point with polar coordinates (2, 45°).

1b. Plot the point with polar coordinates (−1, π).

1c. Plot the point with polar coordinates $\left(-1,\ -\frac{\pi}{2}\right)$.

Draw an angle measuring $-\frac{\pi}{2}$ radians in standard position. Then plot a point 1 unit from the origin (pole) along a line in the opposite direction from the terminal side of the angle. Since the terminal side of the angle points down, this means that we move 2 units up.



1c. Plot the point with polar coordinates $\left(-2,\ -\frac{\pi}{2}\right)$.

Objective #2: Find multiple sets of polar coordinates for a given point.

✔ *Solved Problem #2*

2a. Find a representation of $\left(5,\ \frac{\pi}{4}\right)$ in which r is positive and $2\pi < \theta < 4\pi$.

Add 2π to the angle and do not change r.

$$\left(5,\ \frac{\pi}{4}\right) = \left(5,\ \frac{\pi}{4} + 2\pi\right) = \left(5,\ \frac{\pi}{4} + \frac{8\pi}{4}\right) = \left(5,\ \frac{9\pi}{4}\right)$$

✎ *Pencil Problem #2* ✐

2a. Find a representation of $\left(10,\ \frac{3\pi}{4}\right)$ in which r is positive and $2\pi < \theta < 4\pi$.

2b. Find a representation of $\left(5, \frac{\pi}{4}\right)$ in which r is negative and $0 < \theta < 2\pi$.

Add π to the angle and replace r with $-r$.

$$\left(5, \frac{\pi}{4}\right) = \left(-5, \frac{\pi}{4} + \pi\right) = \left(-5, \frac{\pi}{4} + \frac{4\pi}{4}\right) = \left(-5, \frac{5\pi}{4}\right)$$

2b. Find a representation of $\left(10, \frac{3\pi}{4}\right)$ in which r is negative and $0 < \theta < 2\pi$.

2c. Find a representation of $\left(5, \frac{\pi}{4}\right)$ in which r is positive and $-2\pi < \theta < 0$.

Subtract 2π from the angle and do not change r.

$$\left(5, \frac{\pi}{4}\right) = \left(5, \frac{\pi}{4} - 2\pi\right) = \left(5, \frac{\pi}{4} - \frac{8\pi}{4}\right) = \left(5, -\frac{7\pi}{4}\right)$$

2c. Find a representation of $\left(10, \frac{3\pi}{4}\right)$ in which r is positive and $-2\pi < \theta < 0$.

Objective #3: Convert a point from polar to rectangular coordinates.

✔ *Solved Problem #3*

3a. Find the rectangular coordinates of the point with polar coordinates $(3, \pi)$.

$$x = r\cos\theta = 3\cos\pi = 3(-1) = -3$$
$$y = r\sin\theta = 3\sin\pi = 3(0) = 0$$

The rectangular coordinates are $(-3, 0)$.

✎ *Pencil Problem #3* ✎

3a. Find the rectangular coordinates of the point with polar coordinates $\left(2, \frac{\pi}{3}\right)$.

3b. Find the rectangular coordinates of the point with polar coordinates $\left(-10, \frac{\pi}{6}\right)$.

$$x = r\cos\theta = -10\cos\frac{\pi}{6} = -10\left(\frac{\sqrt{3}}{2}\right) = -5\sqrt{3}$$

$$y = r\sin\theta = -10\sin\frac{\pi}{6} = -10\left(\frac{1}{2}\right) = -5$$

The rectangular coordinates are $(-5\sqrt{3}, \ -5)$.

3b. Find the rectangular coordinates of the point with polar coordinates $\left(-4, \frac{\pi}{2}\right)$.

Objective #4: Convert a point from rectangular to polar coordinates.

✔ *Solved Problem #4*

4a. Find polar coordinates of the point with rectangular coordinates $(1, \ -\sqrt{3})$.

The point $(1, \ -\sqrt{3})$ is in quadrant IV. Find r.

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

Find θ. Since $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ and θ lies in quadrant IV,

$$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}.$$

The polar coordinates are $\left(2, \frac{5\pi}{3}\right)$.

✎ *Pencil Problem #4* ✐

4a. Find polar coordinates of the point with rectangular coordinates (–2, 2).

4b. Find polar coordinates of the point with rectangular coordinates (0, –4).

The point (0, –4) lies on the negative y-axis 4 units below the origin. Thus, $r = 4$ and $\theta = \frac{3\pi}{2}$.

The polar coordinates are $\left(4, \frac{3\pi}{2}\right)$.

4b. Find polar coordinates of the point with rectangular coordinates (5, 0).

Objective #5: Convert an equation from rectangular to polar coordinates.

✔ *Solved Problem #5*

5a. Covert $3x - y = 6$ to a polar equation.

Replace x with $r\cos\theta$ and y with $r\sin\theta$ and solve for r.

$$3x - y = 6$$
$$3r\cos\theta - r\sin\theta = 6$$
$$r(3\cos\theta - \sin\theta) = 6$$
$$r = \frac{6}{3\cos\theta - \sin\theta}$$

✎ *Pencil Problem #5* ✎

5a. Covert $3x + y = 7$ to a polar equation.

5b. Covert $x^2 + (y+1)^2 = 1$ to a polar equation.

Replace x with $r\cos\theta$ and y with $r\sin\theta$ and solve for r.

$$x^2 + (y+1)^2 = 1$$
$$(r\cos\theta)^2 + (r\sin\theta + 1)^2 = 1$$
$$r^2\cos^2\theta + r^2\sin^2\theta + 2r\sin\theta + 1 = 1$$
$$r^2(\cos^2\theta + \sin^2\theta) + 2r\sin\theta = 0$$
$$r^2 + 2r\sin\theta = 0$$
$$r(r + 2\sin\theta) = 0$$
$$r = 0 \quad \text{or} \quad r + 2\sin\theta = 0$$
$$r = -2\sin\theta$$

The graph of $r = 0$ is the pole. The graph of $r = -2\sin\theta$ also includes the pole, so it is not necessary to include the equation $r = 0$.

The polar equation is $r = -2\sin\theta$.

5b. Covert $(x-2)^2 + y^2 = 4$ to a polar equation.

Objective #6: Convert an equation from polar to rectangular coordinates.

✔ *Solved Problem #6*

6a. Convert $r = 4$ to a rectangular equation.

Square each side in anticipation of using $r^2 = x^2 + y^2$.

$$r = 4$$
$$r^2 = 4^2$$
$$x^2 + y^2 = 16$$

The rectangular equation is $x^2 + y^2 = 16$, which we recognize as the equation of a circle centered at the origin with radius 4.

✎ *Pencil Problem #6* ✐

6a. Convert $r = 8$ to a rectangular equation.

6b. Convert $\theta = \dfrac{3\pi}{4}$ to a rectangular equation.

Take the tangent of each side and use $\tan\theta = \dfrac{y}{x}$. Then solve for y.

$$\tan\theta = \tan\frac{3\pi}{4}$$
$$\frac{y}{x} = -1$$
$$y = -x$$

The rectangular equation is $y = -x$, which we recognize as the equation of a line passing through the origin with slope -1.

6b. Convert $\theta = \dfrac{\pi}{2}$ to a rectangular equation.

6c. Convert $r = -2\sec\theta$ to a rectangular equation.

Use a reciprocal identity to rewrite the secant in terms of cosine. Multiply each side by $\cos\theta$ and replace $r\cos\theta$ with x.

$$r = -2\sec\theta$$
$$r = \frac{-2}{\cos\theta}$$
$$r\cos\theta = -2$$
$$x = -2$$

The rectangular equation is $x = -2$, which we recognize as the equation of a vertical line with x-intercept -2.

6c. Convert $r = 4\csc\theta$ to a rectangular equation.

6d. Convert $r = 10\sin\theta$ to a rectangular equation.

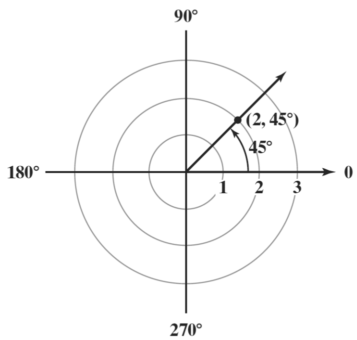
Multiply both sides by r and then replace r^2 with $x^2 + y^2$ and $r\sin\theta$ with y.

$$r = 10\sin\theta$$
$$r^2 = 10r\sin\theta$$
$$x^2 + y^2 = 10y$$
$$x^2 + y^2 - 10y = 0$$
$$x^2 + (y^2 - 10y + 25) = 25$$
$$x^2 + (y-5)^2 = 25$$

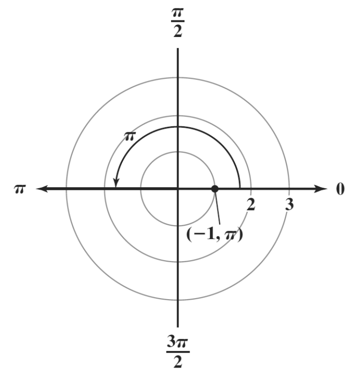
The rectangular equation is $x^2 + (y-5)^2 = 25$, which we recognize as the equation of a circle centered at (0, 5) with radius 5.

6d. Convert $r = 12\cos\theta$ to a rectangular equation.

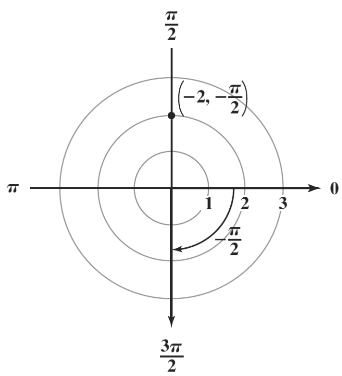
Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:



1a. *(6.3 #11)*



1b. *(6.3 #17)*



1c. *(6.3 #19)*

2a. $\left(10,\ \frac{11\pi}{4}\right)$ **2b.** $\left(-10,\ \frac{7\pi}{4}\right)$ **2c.** $\left(10,\ -\frac{5\pi}{4}\right)$ *(6.3 #23)*

3a. $(1,\ \sqrt{3})$ *(6.3 #35)* **3b.** $(0, -4)$ *(6.3 #37)*

4a. $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$ *(6.3 #41)* **4b.** $(5, 0)$ *(6.3 #47)*

5a. $r = \frac{7}{3\cos\theta + \sin\theta}$ *(6.3 #49)* **5b.** $r = 4\cos\theta$ *(6.3 #55)*

6a. $x^2 + y^2 = 64$ *(6.3 #59)* **6b.** $x = 0$ *(6.3 #61)*

6c. $y = 4$ *(6.3 #65)* **6d.** $(x-6)^2 + y^2 = 36$ *(6.3 #69)*

Section 6.4
Graphs of Polar Equations

Snails, Roses, and Propellers?

In the previous section, we looked at polar equations representing lines and circles that could be easily converted to rectangular form. In this section, we will graph polar equations without converting them. These polar equations will be relatively simple but in some cases extremely difficult to represent or graph in rectangular coordinates. The graphs will have interesting shapes that may remind you of snails, roses, and propellers.

Objective #1: Use point plotting to graph polar equations.

✔ *Solved Problem #1*

1. Graph $r = 4\sin\theta$ with θ in radians. Use multiples of $\frac{\pi}{6}$ from 0 to π to generate coordinates for points.

We begin by creating a table of coordinates for some points on the graph.

θ	$r = 4\sin\theta$	(r, θ)
0	$r = 4\sin 0 = 4\cdot 0 = 0$	$(0, 0)$
$\frac{\pi}{6}$	$r = 4\sin\frac{\pi}{6} = 4\cdot\frac{1}{2} = 2$	$\left(2, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	$r = 4\sin\frac{\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$	$\left(2\sqrt{3}, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	$r = 4\sin\frac{\pi}{2} = 4\cdot 1 = 4$	$\left(4, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	$r = 4\sin\frac{2\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$	$\left(2\sqrt{3}, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	$r = 4\sin\frac{5\pi}{6} = 4\cdot\frac{1}{2} = 2$	$\left(2, \frac{5\pi}{6}\right)$
π	$r = 4\sin\pi = 4\cdot 0 = 0$	$(0, \pi)$

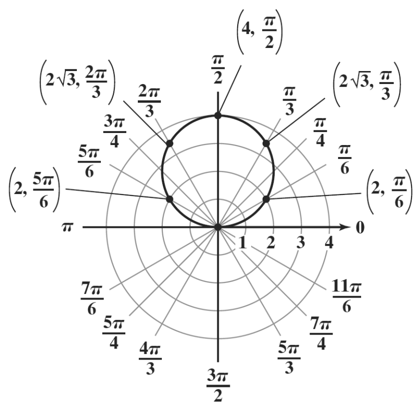
The third column of the table lists the polar coordinates of several points on the graph. Note that the first and last points both represent the pole.

(continued on next page)

✎ *Pencil Problem #1* ✎

1. Graph $r = 2\cos\theta$ with θ in radians. Use multiples of $\frac{\pi}{6}$ from 0 to π to generate coordinates for points.

Plot the points and connect them with a smooth curve. The graph is a circle of radius 2 centered at (0, 2) in rectangular coordinates.



$r = 4\sin\theta$

Objective #2: Use symmetry to graph polar equations.

✔ *Solved Problem #2*

2a. Check for symmetry and then graph the polar equation: $r = 1 + \cos\theta$.

Polar Axis: Replace θ with $-\theta$ and note that $\cos(-\theta) = \cos\theta$ since cosine is an even function.

$r = 1 + \cos(-\theta)$

$r = 1 + \cos\theta$

The result is equivalent to the original equation. The graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ with $-\theta$ and r with $-r$.

$-r = 1 + \cos(-\theta)$

$-r = 1 + \cos\theta$

$r = -1 - \cos\theta$

The result is not equivalent to the original equation. The graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

(continued on next page)

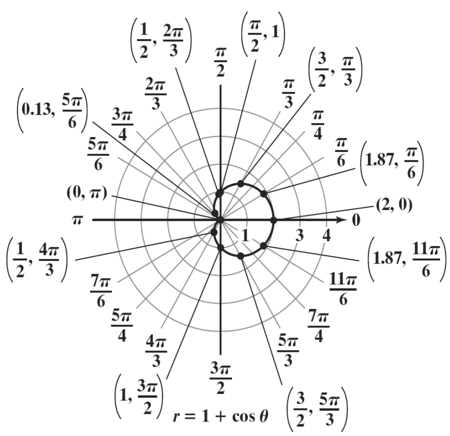
✎ *Pencil Problem #2* ✐

2a. Check for symmetry and then graph the polar equation: $r = 1 - \sin\theta$.

The Pole: Replace r with $-r$.
$-r = 1 + \cos\theta$
$r = -1 - \cos\theta$
The result is not equivalent to the original equation. The graph may or may not be symmetric with respect to the pole.

Because the period of cosine is 2π, we only need to consider values of θ between 0 and 2π. Since we have symmetry with respect to the polar axis, we can further restrict ourselves to values of θ between 0 and π. We will plot a few points, connect them, and then complete the graph by reflecting this portion about the polar axis.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	1.87	$\frac{3}{2}$	1	$\frac{1}{2}$	0.13	0



The shape of this graph is called a cardiod.

2b. Check for symmetry and then graph the polar equation: $r = 1 - 2\sin\theta$.

In the tests below, remember that the sine function is odd, so that $\sin(-\theta) = -\sin\theta$.

Polar Axis: $r = 1 - 2\sin(-\theta)$
$r = 1 + 2\sin\theta$

The Line $\theta = \frac{\pi}{2}$: $-r = 1 - 2\sin(-\theta)$
$-r = 1 + 2\sin\theta$
$r = -1 - 2\sin\theta$

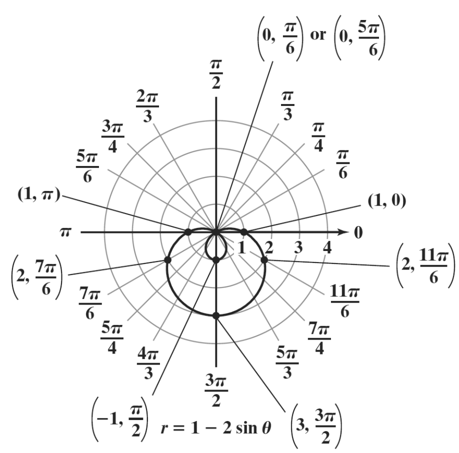
The Pole: $-r = 1 + 2\sin\theta$
$r = -1 - 2\sin\theta$

None of the tests results in an equivalent equation, so the graph may or may not have each of these types of symmetry.

Because we do not have any known symmetry, we must consider values of θ from 0 to 2π.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	1	0	−1	0	1	2	3	2	1

Plot the points and connect them with a smooth curve. Be sure to connect them in the order in which they appear in the table from left to right.



Notice that the graph does have symmetry with respect to the line $\theta = \frac{\pi}{2}$, even though the equation failed the test for this kind of symmetry. This graph is called a limaçon with an inner loop.

2b. Check for symmetry and then graph the polar equation: $r = 1 + 2\cos\theta$.

2c. Check for symmetry and then graph the polar equation: $r = 3\cos 2\theta$.

Polar Axis: $r = 3\cos 2(-\theta)$

$r = 3\cos(-2\theta)$

$r = 3\cos 2\theta$

The Line $\theta = \dfrac{\pi}{2}$: $-r = 3\cos 2(-\theta)$

$-r = 3\cos(-2\theta)$

$-r = 3\cos 2\theta$

$r = -3\cos 2\theta$

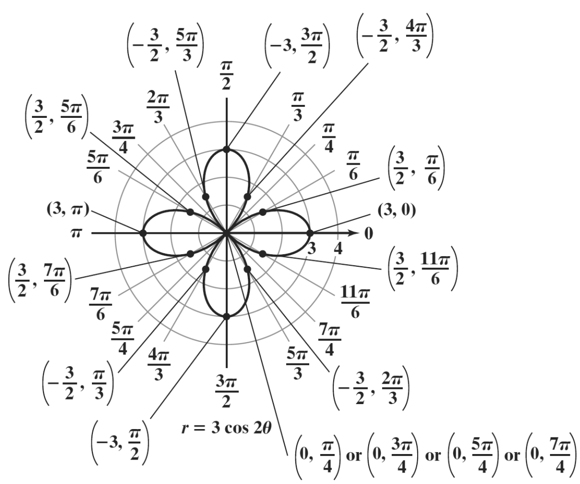
The Pole: $-r = 3\cos 2\theta$

$r = -3\cos 2\theta$

Only the test for symmetry with respect to the polar axis results in an equivalent equation, so the graph has this type of symmetry. It may or may not have the other two types.

We draw the graph for $0 \le \theta \le \pi$ first and then reflect this portion about the polar axis to complete the graph.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	3	$\frac{3}{2}$	0	$-\frac{3}{2}$	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3



Notice that the graph has all three types of symmetry, even though it failed two of the three tests. The shape of the graph is called a rose curve with 4 petals.

2c. Check for symmetry and then graph the polar equation: $r = 4\sin 3\theta$.

2d. Check for symmetry and then graph the polar equation: $r^2 = 4\cos 2\theta.$

Polar Axis: $r^2 = 4\cos 2(-\theta)$

$r^2 = 4\cos(-2\theta)$

$r^2 = 4\cos 2\theta$

The Line $\theta = \dfrac{\pi}{2}$: $(-r)^2 = 4\cos 2(-\theta)$

$r^2 = 4\cos(-2\theta)$

$r^2 = 4\cos 2\theta$

The Pole: $(-r)^2 = 4\cos 2\theta$

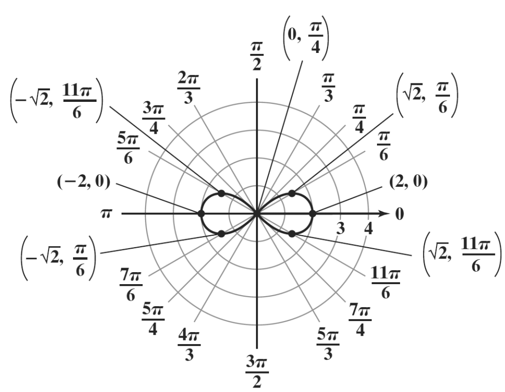
$r^2 = 4\cos 2\theta$

All three tests result in the original equation. The graph has all three types of symmetry. We only need to plot points for $0 \le \theta \le \dfrac{\pi}{2}$. However, notice that for $\dfrac{\pi}{4} < \theta \le \dfrac{\pi}{2}$, $4\cos 2\theta$ is negative and the equation has no solutions. So we only need to consider $0 \le \theta \le \dfrac{\pi}{4}$.

Then we can reflect this portion of the graph about the polar axis and the line $\theta = \dfrac{\pi}{2}$, as necessary.

θ	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

Notice that the entries in the table represent five points, since for two of the angles we have two values for *r*.



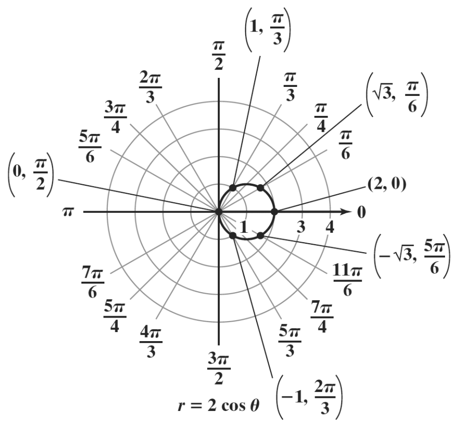
$r^2 = 4\cos 2\theta$

This propeller-shaped graph is called a lemniscate.

2d. Check for symmetry and then graph the polar equation: $r^2 = 9\cos 2\theta.$

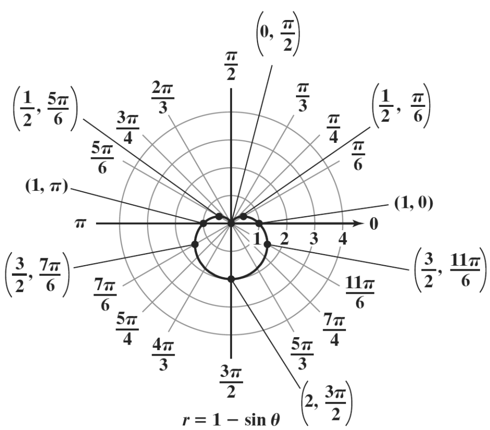
<u>Answers</u> <u>for Pencil Problems</u> <u>*(Textbook Exercise references in parentheses)*</u>:

1.



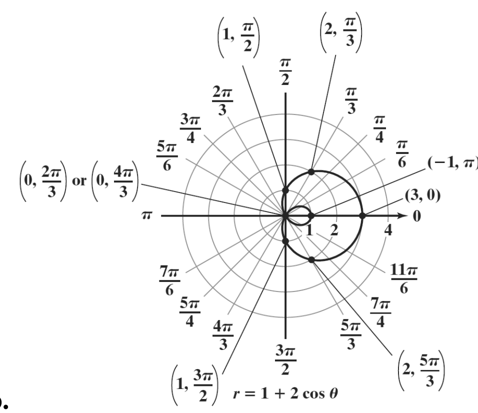
(6.4 #13)

2a.

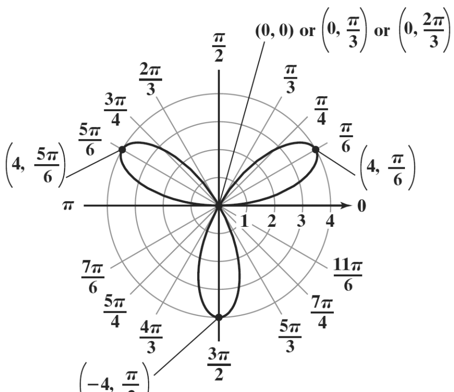


(6.4 #15)

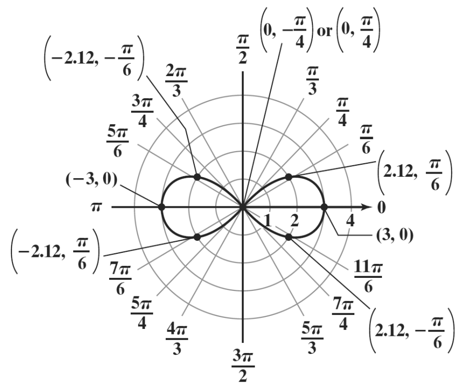
2b.



(6.4 #21)



2c. $r = 4 \sin 3\theta$ *(6.4 #27)*



2d. $r^2 = 9 \cos 2\theta$ *(6.4 #29)*

Section 6.5
Complex Numbers in Polar Form; DeMoivre's Theorem

So We Have Finally Devolved into Chaos?

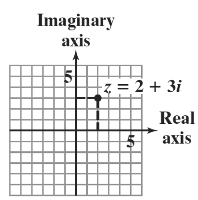
In the previous section, we saw how some complicated graphs had very simple equations in polar coordinates. In this section, we will see how writing complex numbers in polar form makes finding products, quotients, powers, and roots of complex numbers a simple task. In the Exercise Set, you will see how complex numbers are used in the study of seemingly random phenomena that are not actually random at all. Such phenomena are called "chaos" in mathematics.

Objective #1: Plot complex numbers in the complex plane.

✔ *Solved Problem #1*

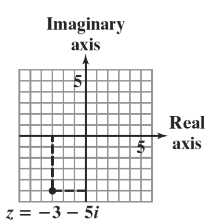
1a. Plot $z = 2 + 3i$ in the complex plane.

Move 2 units to the right along the real axis and then 3 units up parallel to the imaginary axis.



1b. Plot $z = -3 - 5i$ in the complex plane.

Move 3 units to the left along the real axis and then 5 units down parallel to the imaginary axis.



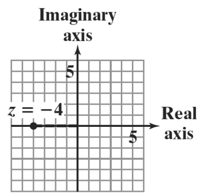
✎ *Pencil Problem #1* ✎

1a. Plot $z = 3 + 2i$ in the complex plane.

1b. Plot $z = -3 + 4i$ in the complex plane.

1c. Plot $z = -4$ in the complex plane.

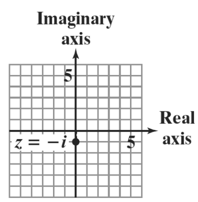
Move 4 units to the left along the real axis.



1c. Plot $z = 3$ in the complex plane.

1d. Plot $z = -i$ in the complex plane.

Move 1 unit down along the imaginary axis.



1d. Plot $z = 4i$ in the complex plane.

Objective #2: Find the absolute value of a complex number.

✔ *Solved Problem #2*

2a. Determine the absolute value of $z = 12 + 5i$.

$$|z| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

The distance from the origin to the point $z = 12 + 5i$ is 13 units.

2b. Determine the absolute value of $z = 2 - 3i$.

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

The distance from the origin to the point $z = 2 - 3i$ is $\sqrt{13}$ units.

✎ *Pencil Problem #2* ✎

2a. Determine the absolute value of $z = -3 + 4i$.

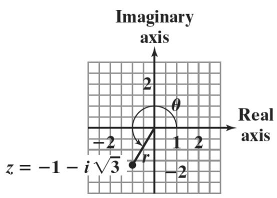
2b. Determine the absolute value of $z = 3 - i$.

Objective #3: Write complex numbers in polar form.

✔ *Solved Problem #3*

3. Plot $z = -1 - i\sqrt{3}$ in the complex plane. Then write z in polar form. Express the argument in radians.

To plot $z = -1 - i\sqrt{3}$, move 1 unit left and then $\sqrt{3} \approx 1.73$ units down



To write the number in polar form, first find the modulus, r. We use $r = \sqrt{a^2 + b^2}$ with $a = -1$ and $b = -\sqrt{3}$.

$$r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

To find θ, we use $\tan\theta = \frac{b}{a}$ with $a = -1$ and $b = -\sqrt{3}$.

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

We know that $\tan\frac{\pi}{3} = \sqrt{3}$ and that θ lies in quadrant III, so

$$\theta = \pi + \frac{\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \frac{4\pi}{3}$$

The polar form of $z = -1 - i\sqrt{3}$ is

$$z = r(\cos\theta + i\sin\theta) = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right).$$

✎ *Pencil Problem #3* ✎

3. Plot $z = 2 + 2i$ in the complex plane. Then write z in polar form. Express the argument in radians.

Objective #4: Convert a complex number from polar to rectangular form.

✔ *Solved Problem #4*

4. Write $z = 4(\cos 30° + i\sin 30°)$ in rectangular form.

We substitute the exact values $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$ into $z = 4(\cos 30° + i\sin 30°)$ and simplify.

$$z = 4(\cos 30° + i\sin 30°) = 4\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\sqrt{3} + 2i$$

✎ *Pencil Problem #4* ✐

4. Write $z = 4(\cos 240° + i\sin 240°)$ in rectangular form.

Objective #5: Find products of complex numbers in polar form.

✔ *Solved Problem #5*

5. Find the product of the complex numbers
$z_1 = 6(\cos 40° + i\sin 40°)$ and
$z_2 = 5(\cos 20° + i\sin 20°)$.
Leave the answer in polar form.

Multiply the moduli and add the arguments.

$$\begin{aligned} z_1 z_2 &= [6(\cos 40° + i\sin 40°)][5(\cos 20° + i\sin 20°)] \\ &= (6 \cdot 5)[\cos(40° + 20°) + i\sin(40° + 20°)] \\ &= 30(\cos 60° + i\sin 60°) \end{aligned}$$

✎ *Pencil Problem #5* ✐

5. Find the product of the complex numbers

$z_1 = 3\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$ and

$z_2 = 4\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$.

Leave the answer in polar form.

Objective #6: Find quotients of complex numbers in polar form.

✔ *Solved Problem #6*

6. Find the quotient $\frac{z_1}{z_2}$ of the complex numbers

$z_1 = 50\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ and

$z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. Leave the answer in polar form.

Divide the moduli and subtract the arguments.

$$\frac{z_1}{z_2} = \frac{50\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}{5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$$
$$= \frac{50}{5}\left[\cos\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right)\right]$$
$$= 10\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right)$$
$$= 10(\cos\pi + i\sin\pi)$$

✎ *Pencil Problem #6* ✎

6. Find the quotient $\frac{z_1}{z_2}$ of the complex numbers

$z_1 = 20(\cos 75° + i\sin 75°)$ and

$z_2 = 4(\cos 25° + i\sin 25°)$. Leave the answer in polar form.

Objective #7: Find powers of complex numbers in polar form.

✔ *Solved Problem #7*

7a. Find $[2(\cos 30° + i\sin 30°)]^5$. Write the answer in rectangular form.

Raise the modulus to the 5th power and multiply the argument by 5. Then replace the trigonometric expressions with exact values and simplify.

$$[2(\cos 30° + i\sin 30°)]^5$$
$$= 2^5[\cos(5\cdot 30°) + i\sin(5\cdot 30°)]$$
$$= 32(\cos 150° + i\sin 150°)$$
$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -16\sqrt{3} + 16i$$

✎ *Pencil Problem #7* ✎

7a. Find $[4(\cos 15° + i\sin 15°)]^3$. Write the answer in rectangular form.

7b. Find $(1+i)^4$ using DeMoivre's Theorem. Write the answer in rectangular form.

First write 1 + *i* in polar form. This number is in the form $z = a + bi$ with a = 1 and b = 1. The modulus is

$$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}.$$

The point lies in quadrant I and

$$\tan\theta = \frac{b}{a} = \frac{1}{1} = 1.$$

We know that $\tan\frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is a quadrant I angle, so $\theta = \frac{\pi}{4}.$

Thus, $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$

Now use DeMoivre's Theorem.

$$\begin{aligned}(1+i)^4 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4 \\ &= (\sqrt{2})^4\left[\cos\left(4\cdot\frac{\pi}{4}\right) + i\sin\left(4\cdot\frac{\pi}{4}\right)\right] \\ &= 4(\cos\pi + i\sin\pi) \\ &= 4(-1 + i\cdot 0) \\ &= -4\end{aligned}$$

7b. Find $(1+i)^5$ using DeMoivre's Theorem. Write the answer in rectangular form.

Objective #8: Find roots of complex numbers in polar form.

✔ *Solved Problem #8*

8. Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$. Write roots in polar form with θ in degrees.

We use DeMoivre's Theorem to find complex roots.

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°\cdot k}{n}\right) + i\sin\left(\frac{\theta + 360°\cdot k}{n}\right)\right] \text{ for}$$
$k = 0, 1, 2, \ldots, n - 1$

The complex number $16(\cos 60° + i\sin 60°)$ is in polar form with $r = 16$ and $\theta = 60°$. Since we are looking for fourth roots, $n = 4$ and we evaluate the formula for $k = 0, 1, 2,$ and 3.

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 0}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 0}{4}\right)\right]$$
$$= 2\left(\cos\frac{60°}{4} + i\sin\frac{60°}{4}\right) = 2(\cos 15° + i\sin 15°)$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 1}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 1}{4}\right)\right]$$
$$= 2\left(\cos\frac{420°}{4} + i\sin\frac{420°}{4}\right) = 2(\cos 105° + i\sin 105°)$$

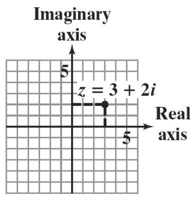
$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 2}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 2}{4}\right)\right]$$
$$= 2\left(\cos\frac{780°}{4} + i\sin\frac{780°}{4}\right) = 2(\cos 195° + i\sin 195°)$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 3}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 3}{4}\right)\right]$$
$$= 2\left(\cos\frac{1140°}{4} + i\sin\frac{1140°}{4}\right) = 2(\cos 285° + i\sin 285°)$$

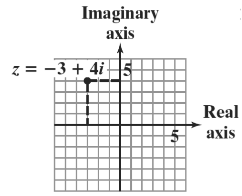
✎ *Pencil Problem #8* ✐

8. Find all the complex cube roots of $8(\cos 210° + i\sin 210°)$. Write roots in polar form with θ in degrees.

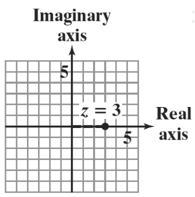
Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 

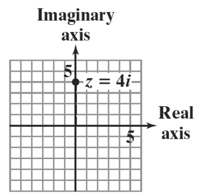
(6.5 #5)

1b. 

(6.5 #9)

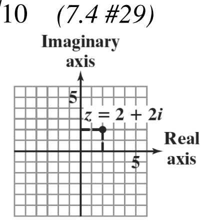
1c. 

. *(6.5 #3)*

1d. 

(6.5 #1)

2a. 5 *(7.4 #29)* **2b.** $\sqrt{10}$ *(7.4 #29)*

3. $2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ 

(6.5 #11)

4. $-2-2i\sqrt{3}$ *(6.5 #29)*

5. $12\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)$ *(6.5 #39)*

6. $5(\cos 50°+i\sin 50°)$ *(6.5 #45)*

7a. $32\sqrt{2}+32i\sqrt{2}$ *(6.5 #53)* **7b.** $-4-4i$ *(6.5 #61)*

8. $z_0=2(\cos 70°+i\sin 70°); z_1=2(\cos 190°+i\sin 190°); z_2=2(\cos 310°+i\sin 310°)$ *(6.5 #67)*

Section 6.6
Vectors

The Force is With You

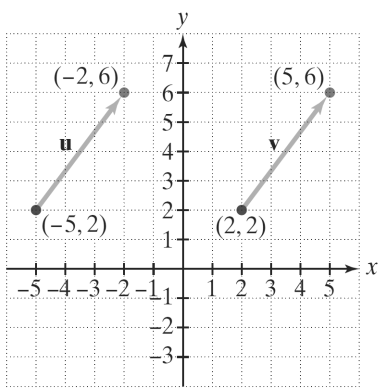
Whether you know it or not, forces are at work all around you. The force of gravity is acting on all objects pulling them downward. To keep an object from falling on the floor, you must apply an equal force in the opposite direction.

In this section, we represent such forces as vectors. We then use these vectors to solve problems involving how much force is required to pull a box up a ramp and how much force is required to keep a car from sliding down a hill.

Objective #1: Use magnitude and direction to show vectors are equal.

✔ *Solved Problem #1*

1. Refer to the figure. Show that **u** = **v**.



Use the distance formula to show that **u** and **v** have the same magnitude.

$$\|\mathbf{u}\| = \sqrt{[-2-(-5)]^2 + (6-2)^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$\|\mathbf{v}\| = \sqrt{(5-2)^2 + (6-2)^2}$$
$$= \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

Both vectors have arrows that point to the upper right. Use the slope formula to show that **u** and **v** have the same direction.

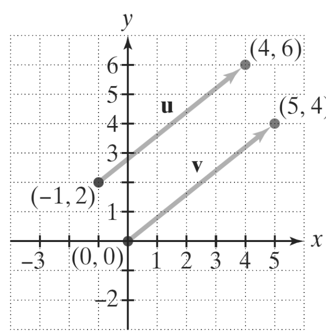
Slope of **u**: $m = \dfrac{6-2}{-2-(-5)} = \dfrac{4}{3}$

Slope of **v**: $m = \dfrac{6-2}{5-2} = \dfrac{4}{3}$

The vectors **u** and **v** have the same magnitude and the same direction, so **u** = **v**.

✎ *Pencil Problem #1* ✎

1. Refer to the figure. Show that **u** = **v**.



Objective #2: Visualize scalar multiplication, vector addition, and vector subtraction as geometric vectors.

✔ Solved Problem #2

2a. True or false: The vector 3**v** is a vector in the same direction as **v** but with 3 times the magnitude of **v**.

True; multiplying a vector by a positive scalar k does not change its direction but does change its magnitude by a factor of k. In symbols, $\|3\mathbf{v}\| = 3\|\mathbf{v}\|$.

2b. True or false: The sum of two vectors, **u** + **v**, can be found by drawing **v** so that it starts where **u** ends and then completing the triangle.

True; when the initial point of **v** coincides with the terminal point of **u**, the vector sum, **u** + **v**, can be drawn with the same initial point as **u** and the same terminal point as **v**, completing a triangle.

✎ Pencil Problem #2 ✎

2a. True or false: The vector −**v** is a vector with the same magnitude as **v** but with opposite direction.

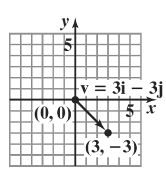
2b. True or false: The difference of two vectors, **u** − **v**, can be found by drawing −**v** so that it starts where **u** ends and then completing the triangle.

Objective #3: Represent vectors in the rectangular coordinate system.

✔ Solved Problem #3

3a. Sketch the vector **v** = 3**i** − 3**j** and find its magnitude.

The vector is in the form **v** = a**i** + b**j** with a = 3 and b = −3. Sketch the vector beginning at the origin, (0, 0), and ending at (a, b) = (3, −3).



$$\|\mathbf{v}\| = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

✎ Pencil Problem #3 ✎

3a. Sketch the vector **v** = 3**i** + **j** and find its magnitude.

3b. Let **v** be the vector from initial point $P_1 = (-1, 3)$ to terminal point $P_2 = (2, 7)$. Write **v** in terms of **i** and **j**.

Let $P_1 = (x_1, y_1) = (-1, 3)$ and $P_2 = (x_2, y_2) = (2, 7)$.

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$
$$= [2-(-1)]\mathbf{i} + (7-3)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$$

3b. Let **v** be the vector from initial point $P_1 = (-4, -4)$ to terminal point $P_2 = (6, 2)$. Write **v** in terms of **i** and **j**.

Objective #4: Perform operations with vectors in terms of **i** and **j**.

✔ *Solved Problem #4*

4a. If $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 5\mathbf{j}$, find $\mathbf{v} + \mathbf{w}$.

$\mathbf{v} + \mathbf{w}$
$= (7\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} - 5\mathbf{j})$
$= (7 + 4)\mathbf{i} + [3 + (-5)]\mathbf{j}$
$= 11\mathbf{i} - 2\mathbf{j}$

4b. If $\mathbf{v} = 7\mathbf{i} + 10\mathbf{j}$, find $-5\mathbf{v}$.

$-5\mathbf{v} = -5(7\mathbf{i} + 10\mathbf{j}) = -35\mathbf{i} - 50\mathbf{j}$

4c. If $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 5\mathbf{j}$, find $6\mathbf{v} - 3\mathbf{w}$.

$6\mathbf{v} - 3\mathbf{w}$
$= 6(7\mathbf{i} + 3\mathbf{j}) - 3(4\mathbf{i} - 5\mathbf{j})$
$= 42\mathbf{i} + 18\mathbf{j} - 12\mathbf{i} + 15\mathbf{j}$
$= (42 - 12)\mathbf{i} + (18 + 15)\mathbf{j}$
$= 30\mathbf{i} + 33\mathbf{j}$

✎ *Pencil Problem #4* ✐

4a. If $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j}$, find $\mathbf{u} + \mathbf{v}$.

4b. If $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j}$, find $5\mathbf{v}$.

4c. If $\mathbf{v} = -3\mathbf{i} + 7\mathbf{j}$ and $\mathbf{w} = -\mathbf{i} - 6\mathbf{j}$, find $3\mathbf{v} - 4\mathbf{w}$.

Objective #5: Find the unit vector in the direction of **v**.

✔ *Solved Problem #5*

5. Write a unit vector in the same direction as $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$.

First find the magnitude of **v.**

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Now divide **v** by its magnitude.

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{4\mathbf{i} - 3\mathbf{j}}{5} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$$

✎ *Pencil Problem #5* ✎

5. Write a unit vector in the same direction as $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Objective #6: Write a vector in terms of its magnitude and direction.

✔ *Solved Problem #6*

6. The jet stream is blowing at 60 miles per hour in the direction of N45°E. Express its velocity as a vector **v** in terms of **i** and **j**.

Since the direction of the jet stream forms a 45° angle with a north-south line on the east side of the north-south line, the vector's direction angle is $\theta = 90° - 45° = 45°$.

Since the jet stream is blowing at 60 miles per hour, the magnitude of **v** is 60: $\|\mathbf{v}\| = 60$.

Use the formula for a vector in terms of magnitude and direction.

$$\begin{aligned}\mathbf{v} &= \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j} \\ &= 60\cos 45°\mathbf{i} + 60\sin 45°\mathbf{j} \\ &= 60\left(\frac{\sqrt{2}}{2}\right)\mathbf{i} + 60\left(\frac{\sqrt{2}}{2}\right)\mathbf{j} \\ &= 30\sqrt{2}\mathbf{i} + 30\sqrt{2}\mathbf{j}\end{aligned}$$

✎ *Pencil Problem #6* ✎

6. A quarterback releases a football with a sped of 44 feet per second at an angle of 30° with the horizontal. Express its velocity as a vector **v** in terms of **i** and **j**.

Objective #7: Solve applied problems involving vectors.

✔ Solved Problem #7

7. Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, of magnitude 30 and 60 pounds, respectively, act on an object. The direction of $\mathbf{F}_1$ is N10°E and the direction of $\mathbf{F}_2$ is N60°E. Find the magnitude, to the nearest hundredth of a pound, and the direction angle, to the nearest tenth of a degree, of the resultant force.

The direction angle for $\mathbf{F}_1$ is $90° - 10° = 80°$, and the direction angle for $\mathbf{F}_2$ is $90° - 60° = 30°$.

$$\mathbf{F}_1 = 30\cos 80°\mathbf{i} + 30\sin 80°\mathbf{j} \approx 5.21\mathbf{i} + 29.54\mathbf{j}$$
$$\mathbf{F}_2 = 60\cos 30°\mathbf{i} + 60\sin 30°\mathbf{j} \approx 51.96\mathbf{i} + 30\mathbf{j}$$

The resultant force is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &\approx (5.21\mathbf{i} + 29.54\mathbf{j}) + (51.96\mathbf{i} + 30\mathbf{j})\\ &= (5.21 + 51.96)\mathbf{i} + (29.54 + 30)\mathbf{j}\\ &= 57.17\mathbf{i} + 59.54\mathbf{j}.\end{aligned}$$

Its magnitude is

$$\|\mathbf{F}\| = \sqrt{57.17^2 + 59.54^2} \approx 82.54 \text{ pounds.}$$

To find the direction angle, use $\cos\theta = \dfrac{a}{\|\mathbf{F}\|}$, where $\mathbf{F} = a\mathbf{i} + b\mathbf{j} = 57.17\mathbf{i} + 59.54\mathbf{j}$, so $a = 57.17$.

$$\cos\theta = \frac{a}{\|\mathbf{F}\|} = \frac{57.17}{82.54}$$

So, the direction angle is $\theta = \cos^{-1}\dfrac{57.17}{82.54} \approx 46.2°$.

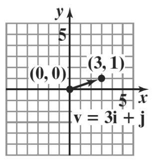
✎ Pencil Problem #7 ✎

7. Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, of magnitude 70 and 50 pounds, respectively, act on an object. The direction of $\mathbf{F}_1$ is S56°E and the direction of $\mathbf{F}_2$ is N72°E. Find the magnitude, to the nearest hundredth of a pound, and the direction angle, to the nearest tenth of a degree, of the resultant force.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. Magnitude: $\|\mathbf{u}\| = \|\mathbf{v}\| = \sqrt{41}$; Direction: Both vectors have arrows that point to the upper right and slopes of $\frac{4}{5}$.
The vectors **u** and **v** have the same magnitude and the same direction, so **u** = **v**. *(6.6 #1)*

2a. true **2b.** true

3a. $\|\mathbf{v}\| = \sqrt{10}$ 

(6.6 #5)

3b. $10\mathbf{i} + 6\mathbf{j}$ *(6.6 #13)*

4a. $-\mathbf{i} + 2\mathbf{j}$ *(6.6 #21)* **4b.** $-15\mathbf{i} + 35\mathbf{j}$ *(6.6 #27)* **4c.** $-5\mathbf{i} + 45\mathbf{j}$ *(6.6 #33)*

5. $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$ *(6.6 #41)*

6. $\mathbf{v} = 22\sqrt{3}\mathbf{i} + 22\mathbf{j}$ *(6.6 #65)*

7a. magnitude: ≈108.21 pounds; direction angle: ≈347.4° *(6.6 #71)*

Section 6.7
The Dot Product

Let's Get to Work!

In this section, you will learn how to find the dot product of two vectors. The dot product is very easy to compute and has many important applications, including computing work. When the force moving an object is applied in the direction of the movement, the work done is a simple product of two real numbers, the magnitude of the force and the distance the object is moved. However, when the force is applied at angle to the direction of the movement, we will use the dot product of two vectors to compute work. So, let's get ready to work!

Objective #1: Find the dot product of two vectors.

✔ ***Solved Problem #1***	✎ ***Pencil Problem #1***
1a. If $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, find $\mathbf{v} \cdot \mathbf{w}$. $\mathbf{v} \cdot \mathbf{w} = 7(2) + (-4)(-1) = 14 + 4 = 18$	**1a.** If $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$, find $\mathbf{v} \cdot \mathbf{w}$.
1b. If $\mathbf{v} = 7\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, find $\mathbf{w} \cdot \mathbf{v}$. $\mathbf{w} \cdot \mathbf{v} = 2(7) + (-1)(-4) = 14 + 4 = 18$	**1b.** If $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$, find $\mathbf{w} \cdot \mathbf{v}$.
1c. If $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$, find $\mathbf{w} \cdot \mathbf{w}$. $\mathbf{w} \cdot \mathbf{w} = 2(2) + (-1)(-1) = 4 + 1 = 5$	**1c.** If $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j}$, find $\mathbf{v} \cdot \mathbf{v}$.

Objective #2: Find the angle between two vectors.

✔ *Solved Problem #2*

2. Find the angle between the vectors $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$. Round to the nearest tenth of a degree.

To use the formula for the angle between two vectors, $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$, we need to know $\mathbf{v} \cdot \mathbf{w}$, $\|\mathbf{v}\|$, and $\|\mathbf{w}\|$.

$$\mathbf{v} \cdot \mathbf{w} = 4(1) + (-3)(2) = 4 - 6 = -2$$
$$\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$
$$\|\mathbf{w}\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$$

Now, use the formula.

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-2}{5 \cdot \sqrt{5}} = -\frac{2}{5\sqrt{5}}$$

So, the angle between the vectors is $\theta = \cos^{-1}\left(-\dfrac{2}{5\sqrt{5}}\right) \approx 100.3°$.

✎ *Pencil Problem #2* ✎

2. Find the angle between the vectors $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - \mathbf{j}$. Round to the nearest tenth of a degree.

Objective #3: Use the dot product to determine if two vectors are orthogonal.

✔ *Solved Problem #3*

3. Are the vectors $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 4\mathbf{j}$ orthogonal?

Find the dot product.

$$\mathbf{v} \cdot \mathbf{w} = 2(6) + 3(-4) = 12 - 12 = 0$$

Since the dot product is 0, the vectors are orthogonal.

✎ *Pencil Problem #3* ✎

3. Are the vectors $\mathbf{v} = 2\mathbf{i} + 8\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - \mathbf{j}$ orthogonal?

Objective #4: Find the projection of a vector onto another vector.

✔ *Solved Problem #4*

4. If $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$, find the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

To use the formula for vector projection,

$\text{proj}_{\mathbf{w}}\mathbf{v} = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$, we need to know $\mathbf{v} \cdot \mathbf{w}$ and $\|\mathbf{w}\|$.

$$\mathbf{v} \cdot \mathbf{w} = 2(1) + (-5)(-1) = 2 + 5 = 7$$

$$\|\mathbf{w}\| = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Now, use the formula.

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w} = \frac{7}{(\sqrt{2})^2}\mathbf{w} = \frac{7}{2}(\mathbf{i} - \mathbf{j}) = \frac{7}{2}\mathbf{i} - \frac{7}{2}\mathbf{j}$$

✎ *Pencil Problem #4* ✎

4. If $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$, find the vector projection of $\mathbf{v}$ onto $\mathbf{w}$.

Objective #5: Express a vector as the sum of two orthogonal vectors.

✔ *Solved Problem #5*

5. Let $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. (These are the vectors from Solved Problem #4.) Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

The vector $\mathbf{v}_1$ is the vector projection of $\mathbf{v}$ onto $\mathbf{w}$, which we found for these two vectors in Solved Problem #4. Subtract $\mathbf{v}_1$ from $\mathbf{v}$ to find $\mathbf{v}_2$.

$$\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{7}{2}\mathbf{i} - \frac{7}{2}\mathbf{j}$$

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1 = (2\mathbf{i} - 5\mathbf{j}) - \left(\frac{7}{2}\mathbf{i} - \frac{7}{2}\mathbf{j}\right) = -\frac{3}{2}\mathbf{i} - \frac{3}{2}\mathbf{j}$$

✎ *Pencil Problem #5* ✎

5. Let $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + 5\mathbf{j}$. (These are the vectors from Pencil Problem #4.) Decompose $\mathbf{v}$ into two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{w}$.

Objective #6: Compute work.

✔ *Solved Problem #6*

6. A child pulls a wagon along level ground by exerting a force of 20 pounds on a handle that makes an angle of 30° with the horizontal. How much work is done pulling the wagon 150 feet? Round to the nearest foot-pound.

Use the formula $W = \|\mathbf{F}\|\|\overrightarrow{AB}\|\cos\theta,$ where $\|\mathbf{F}\| = 20$ is the magnitude of the force, $\|\overrightarrow{AB}\| = 150$ is the distance the wagon is pulled, and $\theta = 30°$ is the angle between the force and the direction of the motion.

$$W = \|\mathbf{F}\|\|\overrightarrow{AB}\|\cos\theta = (20)(150)\cos 30° \approx 2598$$

The work done is approximately 2598 foot-pounds.

✎ *Pencil Problem #6* ✐

6. A wagon is pulled along level ground by exerting a force of 40 pounds on a handle that makes an angle of 32° with the horizontal. How much work is done pulling the wagon 100 feet? Round to the nearest foot-pound.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. −6 **1b.** −6 **1c.** 41 *(6.7 #3)* **2.** ≈ 160.3° *(6.7 #19)*
3. Yes, they are orthogonal, since their dot product is 0. *(6.7 #25)*
4. $\text{proj}_{\mathbf{w}}\mathbf{v} = -\frac{26}{29}\mathbf{i} + \frac{65}{29}\mathbf{j}$ *(6.7 #35)* **5.** $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v} = -\frac{26}{29}\mathbf{i} + \frac{65}{29}\mathbf{j}$; $\mathbf{v}_2 = \frac{55}{29}\mathbf{i} + \frac{22}{29}\mathbf{j}$ *(6.7 #35)*
6. ≈3392 foot-pounds *(6.7 #55)*