Section 5.1
Verifying Trigonometric Identities

Do you Enjoy Solving Puzzles?

We have already established some basic relationships among the trigonometric functions. The reciprocal, quotient, and Pythagorean identities follow easily from the definitions of the trigonometric functions. The even-odd identities are established using properties of the Cartesian coordinate system.
In this section, we see how new identities can be verified using identities that we already know. The process will involve some trial and error and may remind you of solving a puzzle.

Objective #1: Use fundamental trigonometric identities to verify identities.

✔ *Solved Problem #1*

1a. Verify the identity: $\csc x \tan x = \sec x$.

In many cases, it is helpful to rewrite all the trigonometric functions on one side in terms of sines and cosines using reciprocal and quotient identities. In general, we start by working with the more complicated side, so we begin with the left side and rewrite it in terms of sines and cosines using the reciprocal identity $\csc x = \frac{1}{\sin x}$ and the quotient identity $\tan x = \frac{\sin x}{\cos x}$.

$$\csc x \tan x = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} = \frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x} = \frac{1}{\cos x} = \sec x$$

Note the use of the reciprocal identity $\sec x = \frac{1}{\cos x}$ in the last step. Also note how we started with the left side of the given identity, $\csc x \tan x = \sec x$, and ended with the right side.

✎ *Pencil Problem #1* ✐

1a. Verify the identity: $\sin x \sec x = \tan x$.

1b. Verify the identity: $\cos x \cot x + \sin x = \csc x.$

Once again we will rewrite all the trigonometric functions on one side in terms of sines and cosines. We start by working with the more complicated side, the left side, and rewrite it using the quotient identity $\cot x = \dfrac{\cos x}{\sin x}.$

$$\begin{aligned}\cos x \cot x + \sin x &= \frac{\cos x}{1}\cdot\frac{\cos x}{\sin x} + \sin x\\ &= \frac{\cos^2 x}{\sin x} + \frac{\sin x}{1}\cdot\frac{\sin x}{\sin x}\\ &= \frac{\cos^2 x}{\sin x} + \frac{\sin^2 x}{\sin x}\\ &= \frac{\cos^2 x + \sin^2 x}{\sin x}\\ &= \frac{1}{\sin x}\\ &= \csc x\end{aligned}$$

Note how we expressed both terms on the left side as fractions with a common denominator and then added; this technique is often helpful when one side of the given identity has two terms and the other side has only one term. Also note the use of the Pythagorean identity $\cos^2 x + \sin^2 x = 1.$

1b. Verify the identity: $\csc\theta - \sin\theta = \cot\theta\cos\theta.$ [Hints: The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ can also be used in the form $1 - \sin^2\theta = \cos^2\theta$ and a fraction of the form $\dfrac{a^2}{b}$ can be rewritten as $\dfrac{a}{b}\cdot\dfrac{a}{1}.$]

1c. Verify the identity: $\sin x - \sin x \cos^2 x = \sin^3 x.$

Sometimes factoring out a common factor is helpful.

We start by working with the more complicated side, the left side, and factor out the common factor $\sin x.$

Then we will use the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ in the form $1 - \cos^2 x = \sin^2 x.$

$$\begin{aligned}\sin x - \sin x\cos^2 x &= \sin x(1 - \cos^2 x)\\ &= \sin x \cdot \sin^2 x = \sin^3 x\end{aligned}$$

1c. Verify the identity by using factoring first: $\sec x - \sec x \sin^2 x = \cos x.$ [Hint: After factoring out the common factor on the left side and using a Pythagorean identity as we did in Solved Problem #1c, rewrite secant in terms of cosine.]

1d. Verify the identity: $\dfrac{1+\cos\theta}{\sin\theta}=\csc\theta+\cot\theta.$

There is usually more than one way to verify an identity. Here we will verify the identity in two ways. In the first method, we will start with the left side and work toward the right. In the second method, we will start with the right side and work toward the left.

Method 1: Start with the left side and write the single fraction as the sum of two fractions. Then use appropriate reciprocal and quotient identities.

$$\frac{1+\cos\theta}{\sin\theta}=\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=\csc\theta+\cot\theta$$

Method 2: Start with the right side and rewrite the given functions in terms of sines and cosines. Then add the resulting fractions.

$$\csc\theta+\cot\theta=\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}=\frac{1+\cos\theta}{\sin\theta}$$

Although the equation in Method 2 could be obtained by reversing the equation in Method 1, the thought processes are different. Perhaps one direction is more obvious to you than the other.

1d. Verify the identity in two different ways as in Solved Problem #1d: $\dfrac{1-\sin\theta}{\cos\theta}=\sec\theta-\tan\theta.$

1e. Verify the identity: $\dfrac{\sin x}{1+\cos x}+\dfrac{1+\cos x}{\sin x}=2\csc x.$

We will work with the left side and begin by getting a common denominator so that we can combine the two fractions into one fraction. The LCD is $(1+\cos x)\sin x.$ Watch for opportunities to factor and simplify within the fraction and to apply basic identities. The Pythagorean identity $\cos^2 x+\sin^2 x=1$ will be used to replace $\cos^2 x+\sin^2 x$ with 1 in a key step along the way.

(continued on next page)

1e. Verify the identity: $\dfrac{\cos x}{1-\sin x}+\dfrac{1-\sin x}{\cos x}=2\sec x.$

$$\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}=\frac{\sin x}{1+\cos x}\cdot\frac{\sin x}{\sin x}+\frac{1+\cos x}{\sin x}\cdot\frac{1+\cos x}{1+\cos x}$$
$$=\frac{\sin^2 x}{(1+\cos x)\sin x}+\frac{(1+\cos x)^2}{(1+\cos x)\sin x}$$
$$=\frac{\sin^2 x+(1+\cos x)^2}{(1+\cos x)\sin x}$$
$$=\frac{\sin^2 x+1+2\cos x+\cos^2 x}{(1+\cos x)\sin x}$$
$$=\frac{(\sin^2 x+\cos^2 x)+1+2\cos x}{(1+\cos x)\sin x}$$
$$=\frac{1+1+2\cos x}{(1+\cos x)\sin x}$$
$$=\frac{2+2\cos x}{(1+\cos x)\sin x}$$
$$=\frac{2(1+\cos x)}{(1+\cos x)\sin x}$$
$$=\frac{2(\cancel{1+\cos x})}{(\cancel{1+\cos x})\sin x}$$
$$=\frac{2}{\sin x}$$
$$=2\csc x$$

<u>Answers</u> for Pencil Problems <u>*(Textbook Exercise references in parentheses)*</u>:

1a. $\sin x\sec x=\frac{\sin x}{1}\cdot\frac{1}{\cos x}=\frac{\sin x}{\cos x}=\tan x$ *(5.1 #1)*

1b. $\csc\theta-\sin\theta=\frac{1}{\sin\theta}-\frac{\sin\theta}{1}\cdot\frac{\sin\theta}{\sin\theta}=\frac{1}{\sin\theta}-\frac{\sin^2\theta}{\sin\theta}=\frac{1-\sin^2\theta}{\sin\theta}=\frac{\cos^2\theta}{\sin\theta}=\frac{\cos\theta}{\sin\theta}\cdot\frac{\cos\theta}{1}=\cot\theta\cos\theta.$ *(5.1 #11)*

1c. $\sec x-\sec x\sin^2 x=\sec x(1-\sin^2 x)=\sec x\cos^2 x=\frac{1}{\cos x}\cdot\cos^2 x=\cos x$ *(5.1 #7)*

1d. $\frac{1-\sin\theta}{\cos\theta}=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}=\sec\theta-\tan\theta;\ \sec\theta-\tan\theta=\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}=\frac{1-\sin\theta}{\cos\theta}$ *(5.1 #24)*

1e. $\frac{\cos x}{1-\sin x}+\frac{1-\sin x}{\cos x}=\frac{\cos x\cdot\cos x}{(1-\sin x)\cos x}+\frac{(1-\sin x)(1-\sin x)}{(1-\sin x)\cos x}=\frac{\cos^2 x+(1-\sin x)^2}{(1-\sin x)\cos x}=\frac{\cos^2 x+1-2\sin x+\sin^2 x}{(1-\sin x)\cos x}$

$=\frac{(\cos^2 x+\sin^2 x)+1-2\sin x}{(1-\sin x)\cos x}=\frac{1+1-2\sin x}{(1-\sin x)\cos x}=\frac{2-2\sin x}{(1-\sin x)\cos x}=\frac{2(1-\sin x)}{(1-\sin x)\cos x}=\frac{2}{\cos x}=2\sec x$ *(5.1#31)*

Section 5.2
Sum and Difference Formulas

More Identities?

In this section, you will learn to use another set of identities, known as the sum and difference formulas. The first of these is established using the definitions of the trigonometric functions and the properties of the Cartesian plane. Once the first formula is established, the other three can be verified using it and other basic identities.
Once verified, these formulas can be used to find function values as well as verify other identities. In the Exercise Set, you will see how these formulas can be used to simplify functions that model sound vibrations.

Objective #1: Use the formula for the cosine of the difference of two angles.

✔ *Solved Problem #1*

1a. We know that $\cos 30° = \frac{\sqrt{3}}{2}$. Use the formula for the cosine of the difference of two angles to find the exact value of $\cos 30° = \cos(90° - 60°)$.

The formula is
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

In $\cos(90° - 60°)$, $\alpha = 90°$ and $\beta = 60°$. Apply the formula above, substitute exact values for the four resulting trigonometric expressions, and simplify.

$$\cos(90° - 60°) = \cos 90° \cos 60° + \sin 90° \sin 60°$$
$$= 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = 0 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Note that this result is the same as the known value of $\cos 30° = \frac{\sqrt{3}}{2}$, as it should be.

✎ *Pencil Problem #1* ✎

1a. Use the formula for the cosine of the difference of two angles to find the exact value of $\cos 15° = \cos(45° - 30°)$.

1b. Use the formula for the cosine of the difference of two angles to find the exact value of $\cos 70° \cos 40° + \sin 70° \sin 40°$.

The formula is $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

Notice that $\cos 70° \cos 40° + \sin 70° \sin 40°$ looks like the right side of the formula with $\alpha = 70°$ and $\beta = 40°$.

$$\begin{aligned}\cos 70° \cos 40° + \sin 70° \sin 40° &= \sin(70° - 40°) \\ &= \sin 30° \\ &= \frac{1}{2}\end{aligned}$$

1c. Verify the identity: $\dfrac{\cos(\alpha - \beta)}{\cos\alpha \cos\beta} = 1 + \tan\alpha \tan\beta$.

We begin with the left side and apply the formula for the cosine of the difference of two angles in the numerator. We then rewrite the fraction as the sum of two fractions and simplify.

$$\begin{aligned}\frac{\cos(\alpha - \beta)}{\cos\alpha \cos\beta} &= \frac{\cos\alpha \cos\beta + \sin\alpha \sin\beta}{\cos\alpha \cos\beta} \\ &= \frac{\cos\alpha \cos\beta}{\cos\alpha \cos\beta} + \frac{\sin\alpha \sin\beta}{\cos\alpha \cos\beta} \\ &= 1 + \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\sin\beta}{\cos\beta} \\ &= 1 + \tan\alpha \tan\beta\end{aligned}$$

1b. Use the formula for the cosine of the difference of two angles to find the exact value of $\cos 50° \cos 20° + \sin 50° \sin 20°$.

1c. Verify the identity: $\dfrac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \tan\alpha + \cot\beta$.

Objective #2: Use sum and difference formulas for cosines and sines.

✔ Solved Problem #2

2. We are given that $\sin\alpha = \frac{4}{5}$ for a quadrant II angle α and $\sin\beta = \frac{1}{2}$ for a quadrant I angle β. Use this information in Solved Problems #2a–d.

2a. Find the exact value of $\cos\alpha$.

The value of $\cos\alpha$ is negative, since α is a quadrant II angle. Use the Pythagorean identity $\sin^2\alpha + \cos^2\alpha = 1$ to find $\cos\alpha$, choosing the negative value.

$$\left(\frac{4}{5}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \left(\frac{4}{5}\right)^2 = \frac{25}{25} - \frac{16}{25} = \frac{9}{25}$$

$$\cos\alpha = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

2b. Find the exact value of $\cos\beta$.

The value of $\cos\beta$ is positive, since β is a quadrant I angle. Use the Pythagorean identity $\sin^2\beta + \cos^2\beta = 1$ to find $\cos\beta$, choosing the positive value.

$$\left(\frac{1}{2}\right)^2 + \cos^2\beta = 1$$

$$\cos^2\beta = 1 - \left(\frac{1}{2}\right)^2 = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

$$\cos\beta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

✎ Pencil Problem #2 ✎

2. We are given that $\sin\alpha = \frac{3}{5}$ for a quadrant I angle α and $\sin\beta = \frac{5}{13}$ for a quadrant II angle β. Use this information in Pencil Problems #2a–d.

2a. Find the exact value of $\cos\alpha$.

2b. Find the exact value of $\cos\beta$.

2c. Find the exact value of $\cos(\alpha+\beta)$.

We use the sum formula for cosine and substitute the given values, $\sin\alpha = \frac{4}{5}$ and $\sin\beta = \frac{1}{2}$, as well as the values we just found, $\cos\alpha = -\frac{3}{5}$ and $\cos\beta = \frac{\sqrt{3}}{2}$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= -\frac{3}{5}\cdot\frac{\sqrt{3}}{2} - \frac{4}{5}\cdot\frac{1}{2} \\ &= -\frac{3\sqrt{3}}{10} - \frac{4}{10} = \frac{-3\sqrt{3}-4}{10}\end{aligned}$$

2c. Find the exact value of $\cos(\alpha+\beta)$.

2d. Find the exact value of $\sin(\alpha+\beta)$.

We use the sum formula for sine and substitute the values $\sin\alpha = \frac{4}{5}$, $\sin\beta = \frac{1}{2}$, $\cos\alpha = -\frac{3}{5}$, and $\cos\beta = \frac{\sqrt{3}}{2}$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{4}{5}\cdot\frac{\sqrt{3}}{2} + \left(-\frac{3}{5}\right)\cdot\frac{1}{2} \\ &= \frac{4\sqrt{3}}{10} - \frac{3}{10} = \frac{4\sqrt{3}-3}{10}\end{aligned}$$

2d. Find the exact value of $\sin(\alpha+\beta)$.

Objective #3: Use sum and difference formulas for tangents.

✔ Solved Problem #3

3. Verify the identity: $\tan(x+\pi) = \tan x$.

Start with the left side and apply the sum formula for tangent. Use the fact that $\tan\pi = 0$.

$$\begin{aligned}\tan(x+\pi) &= \frac{\tan x + \tan\pi}{1-\tan x\tan\pi}\\ &= \frac{\tan x + 0}{1-\tan x\cdot 0}\\ &= \frac{\tan x}{1}\\ &= \tan x\end{aligned}$$

✎ Pencil Problem #3 ✐

3. Verify the identity: $\tan(2\pi - x) = -\tan x$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ *(5.2 #1)*

1b. $\dfrac{\sqrt{3}}{2}$ *(5.2 #5)*

1c. $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}=\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\sin\beta}=\dfrac{\cancel{\cos\alpha}\cos\beta}{\cancel{\cos\alpha}\sin\beta}+\dfrac{\sin\alpha\cancel{\sin\beta}}{\cos\alpha\cancel{\sin\beta}}=\cot\beta+\tan\alpha=\tan\alpha+\cot\beta$ *(5.2 #9)*

2a. $\dfrac{4}{5}$ **2b.** $-\dfrac{12}{13}$ **2c.** $-\dfrac{63}{65}$ *(5.2 #57a)* **2d.** $-\dfrac{16}{65}$ *(5.2 #57b)*

3. $\tan(2\pi-x)=\dfrac{\tan 2\pi-\tan x}{1+\tan 2\pi\tan x}=\dfrac{0-\tan x}{1+0\cdot\tan x}=\dfrac{-\tan x}{1}=-\tan x$ *(5.2 #37)*

Section 5.3
Double Angle, Power-Reducing, and Half-Angle Formulas

How Far Can You Throw That?

When you throw an object, the distance that the object will travel before hitting the ground depends on the initial speed of the object as well as the angle at which the object leaves your hand. This distance can be modeled by a formula that involves both sines and cosines.
In the Exercise Set, you will see how a trigonometric identity introduced in this section can be used to rewrite a formula involving two trigonometric functions as a formula involving only one such function. The simpler form can be used to determine an angle that maximizes throwing distance.

Objective #1: Use the double-angle formulas.

✔ *Solved Problem #1*

1a. If $\sin\theta = \frac{4}{5}$ and θ lies in quadrant II, find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

The formulas $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ require that we know both $\sin\theta$ and $\cos\theta$. We are given that $\sin\theta = \frac{4}{5}$. We need to find $\cos\theta$. Since θ lies in quadrant II, the value of $\cos\theta$ is negative. Use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to find $\cos\theta$, choosing the negative value.

$$\left(\frac{4}{5}\right)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{25}{25} - \frac{16}{25} = \frac{9}{25}$$

$$\cos\theta = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$$

✎ *Pencil Problem #1* ✎

1a. If $\sin\theta = \frac{15}{17}$ and θ lies in quadrant II, find the exact value of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

Now substitute the values into the formulas.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

The formula $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ requires that we know $\tan\theta$. Since we know $\sin\theta = \dfrac{4}{5}$ and $\cos\theta = -\dfrac{3}{5}$, we can use a quotient identity to find $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{5}\cdot\left(-\frac{5}{3}\right) = -\frac{4}{3}$$

Now we can use this value in the formula.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$$
$$= \frac{-\frac{8}{3}}{1-\frac{16}{9}} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = -\frac{8}{3}\cdot\left(-\frac{9}{7}\right) = \frac{24}{7}$$

Note that we can also find $\tan 2\theta$ using the values of $\sin 2\theta$ and $\cos 2\theta$ and a quotient identity.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

1b. Find the exact value of $\cos^2 15° - \sin^2 15°$.

The given expression looks like the right side of the double-angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ with $\theta = 15°$. We use the formula to rewrite the expression and then evaluate.

$$\cos^2 15° - \sin^2 15° = \cos(2 \cdot 15°) = \cos 30° = \frac{\sqrt{3}}{2}$$

1b. Find the exact value of $2\sin 15° \cos 15°$.

1c. Verify the identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Notice that the sine on the left side has 3θ as its argument while the ones on the right have just θ, so we need to use one or more identities that allow us to change the argument. The right side may look more complicated, but we will start with the left side because it contains the more complicated argument and we see more opportunities to apply identities.

We first rewrite $\sin 3\theta$ as $\sin(2\theta + \theta)$ and use the sum formula for sine. Then we use double-angle formulas to simplify the occurrences of $\sin 2\theta$ and $\cos 2\theta$ that result from the first step. Since the right side contains only sine, we choose to use the form of the double-angle formula for cosine that expresses $\cos 2\theta$ in terms of sine only. Still keeping in mind that the right side contains only sine, we replace an occurrence of $\cos^2\theta$ with $1 - \sin^2\theta$, using a form of the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

1c. Verify the identity:
$\sin 4t = 4\sin t\cos^3 t - 4\sin^3 t\cos t$.

Objective #2: Use the power-reducing formulas.

✔ *Solved Problem #2*

2. Write an equivalent expression for $\sin^4 x$ that does not contain powers of trigonometric functions greater than 1.

We begin by writing $\sin^4 x$ as $(\sin^2 x)^2$ and applying the power-reducing formula for sine. After some simplification, we will use the power-reducing formula for cosine.

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 \\
&= \left(\frac{1-\cos 2x}{2}\right)^2 \\
&= \frac{1-2\cos 2x+\cos^2 2x}{4} \\
&= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x \\
&= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\left[\frac{1+\cos 2(2x)}{2}\right] \\
&= \frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x \\
&= \frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x
\end{aligned}$$

✎ *Pencil Problem #2* ✐

2. Write an equivalent expression for $\sin^2 x\cos^2 x$ that does not contain powers of trigonometric functions greater than 1.

Objective #3: Use the half-angle formulas.

✔ *Solved Problem #3*

3. Use $\cos 210° = -\frac{\sqrt{3}}{2}$ and a half-angle formula to find the exact value of $\cos 105°$.

Since an angle that measures 105° lies in quadrant II and cosine is negative in quadrant II, we choose the negative sign in the half-angle formula for cosine.

$$\begin{aligned}\cos 105° &= \cos\frac{210°}{2}\\ &= -\sqrt{\frac{1+\cos 210°}{2}}\\ &= -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}}\\ &= -\sqrt{\frac{2-\sqrt{3}}{4}}\\ &= -\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Note that to simplify the fraction $\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}$ we multiplied the numerator and denominator both by 2: $\frac{\left[1+\left(-\frac{\sqrt{3}}{2}\right)\right]\cdot 2}{2\cdot 2} = \frac{1\cdot 2+\left(-\frac{\sqrt{3}}{2}\right)\cdot 2}{4} = \frac{2-\sqrt{3}}{4}.$

✎ *Pencil Problem #3* ✎

3. Use $\cos 315° = \frac{\sqrt{2}}{2}$ and a half-angle formula to find the exact value of $\cos 157.5°$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $\sin 2\theta = -\frac{240}{289}$; $\cos 2\theta = -\frac{161}{289}$; $\tan 2\theta = \frac{240}{161}$ *(5.3 #7)*

1b. $\frac{1}{2}$ *(5.3 #15)*

1c. $\sin 4t = \sin(2\cdot 2t) = 2\sin 2t\cos 2t = 2(2\sin t\cos t)(\cos^2 t - \sin^2 t) = 4\sin t\cos^3 t - 4\sin^3 t\cos t$ *(5.3 #33)*

2. $\frac{1}{8} - \frac{1}{8}\cos 4x$ *(5.3 #37)*

3. $-\frac{\sqrt{2+\sqrt{2}}}{2}$ *(5.3 #41)*

Section 5.4
Product-to-Sum and Sum-to-Product Formulas

Music to Your Ears

Each time you push a button on a touch-tone phone a sound is produced. This sound can be modeled by the sum of two sine functions. In the Exercise Set, you will learn how to play *Mary Had a Little Lamb* on your phone and you will write the sum of sines producing a note as a product.

Objective #1: Use the product-to-sum formulas.

✔ *Solved Problem #1*

1a. Express $\sin 5x \sin 2x$ as a sum or difference.

We use the formula for the product of two sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)].$$

$$\sin 5x\sin 2x = \frac{1}{2}[\cos(5x-2x)-\cos(5x+2x)]$$

$$= \frac{1}{2}[\cos 3x - \cos 7x]$$

✎ *Pencil Problem #1* ✎

1a. Express $\sin 6x \sin 2x$ as a sum or difference.

✔ *Solved Problem #1*

1b. Express $\cos 7x \cos x$ as a sum or difference.

We use the formula for the product of two cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)].$$

$$\cos 7x\cos x = \frac{1}{2}[\cos(7x-x)+\cos(7x+x)]$$

$$= \frac{1}{2}[\cos 6x + \cos 8x]$$

✎ *Pencil Problem #1* ✎

1b. Express $\sin x \cos 2x$ as a sum or difference.

Objective #2: Use the sum-to-product formulas.

✔ Solved Problem #2

2a. Express $\sin 7x + \sin 3x$ as a product.

We use the formula for the sum of two sines:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\begin{aligned}\sin 7x + \sin 3x &= 2\sin\frac{7x+3x}{2}\cos\frac{7x-3x}{2}\\ &= 2\sin\frac{10x}{2}\cos\frac{4x}{2}\\ &= 2\sin 5x\cos 2x\end{aligned}$$

2b. Express $\cos 3x + \cos 2x$ as a product.

We use the formula for the sum of two cosines:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

$$\begin{aligned}\cos 3x + \cos 2x &= 2\cos\frac{3x+2x}{2}\cos\frac{3x-2x}{2}\\ &= 2\cos\frac{5x}{2}\cos\frac{x}{2}\end{aligned}$$

✎ Pencil Problem #2 ✎

2a. Express $\sin 6x + \sin 2x$ as a product.

2b. Express $\cos 4x + \cos 2x$ as a product.

<u>**Answers**</u> <u>for Pencil Problems</u> <u>*(Textbook Exercise references in parentheses)*</u>:

1a. $\frac{1}{2}[\cos 4x - \cos 8x]$ *(5.4 #1)* **1b.** $\frac{1}{2}[\sin 3x - \sin x]$ *(5.4 #5)*

2a. $2\sin 4x\cos 2x$ *(5.4 #9)* **2b.** $2\cos 3x\cos x$ *(5.4 #13)*

Section 5.5
Trigonometric Equations

How Many Solutions are There?

We have seen that trigonometric functions can be used to model cyclic phenomena such as tides, temperatures, and hours of daylight. For example, we have worked with models where we could estimate the number of hours of daylight at a location on a particular day of the year by evaluating the function for a value of the independent variable representing the day.

In this section's Exercise Set, you will solve trigonometric equations to find the dates on which a certain city has 10.5 hours of daylight. In general, the equation has infinitely many solutions, but when we restrict ourselves to one cycle, one year, there are exactly two solutions. In this section, pay attention to whether you are looking for all solutions or just the solutions that are in a given interval.

Objective #1: Find all solutions of a trigonometric equation.

✔ *Solved Problem #1*

1. Find all solutions of $5\sin x = 3\sin x + \sqrt{3}$.

First isolate $\sin x$ on the left side by first subtracting $3\sin x$ from both sides and then dividing both sides by the resulting coefficient of $\sin x$.

$$5\sin x = 3\sin x + \sqrt{3}$$
$$5\sin x - 3\sin x = 3\sin x - 3\sin x + \sqrt{3}$$
$$2\sin x = \sqrt{3}$$
$$\frac{\not{2}\sin x}{\not{2}} = \frac{\sqrt{3}}{2}$$
$$\sin x = \frac{\sqrt{3}}{2}$$

The sine function is positive in quadrants I and II and sine equals $\frac{\sqrt{3}}{2}$ at $\frac{\pi}{3}$ in quadrant I. In quadrant II, sine equals $\frac{\sqrt{3}}{2}$ at $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. In one period of the sine function, there are two solutions of the given equation, $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$.

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✎ *Pencil Problem #1* ✎

1. Find all solutions of $4\sin\theta - 1 = 2\sin\theta$.

We are asked to find all solutions. Because of the periodic property of the sine function, if we add or subtract any integer multiple of 2π to either of the solutions on the previous page, we will get another solution. Thus, the solutions are

$$x = \frac{\pi}{3} + 2n\pi \text{ and } x = \frac{2\pi}{3} + 2n\pi,$$

where n is any integer.

Objective #2: Solve equations with multiple angles.

✔ *Solved Problem #2*

2a. Solve the equation: $\tan 2x = \sqrt{3},\ 0 \le x < 2\pi.$

The equation already has the tangent expression isolated on the left. Notice that the argument, $2x$, is a double-angle. We solve for $2x$ first and then for x. We also note that we are only looking for solutions x that satisfy $0 \le x < 2\pi$. However, we first identify all solutions and then select those in the given interval.

In Solved Problem #1, we worked with sine which has period 2π. In this problem, we are working with tangent which has period π. In one period, tangent equals $\sqrt{3}$ only once, at $\frac{\pi}{3}$. Thus,

$$2x = \frac{\pi}{3} + n\pi, \text{ where } n \text{ is any integer.}$$

Dividing by 2, we find all solutions of the given equation:

$$x = \frac{\pi}{6} + \frac{n\pi}{2}, \text{ where } n \text{ is any integer.}$$

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✎ *Pencil Problem #2* ✎

2a. Solve the equation: $\tan 3x = \frac{\sqrt{3}}{3},\ 0 \le x < 2\pi.$

We want to find the solutions x that satisfy $0 \le x < 2\pi$. For any negative integer n, x is negative and does not satisfy $0 \le x < 2\pi$. For any integer $n \ge 4$, $x \ge 2\pi$ and does not satisfy $0 \le x < 2\pi$. Here are the values of x for $n = 0$, 1, 2, and 3:

$$x = \frac{\pi}{6} + \frac{0 \cdot \pi}{2} = \frac{\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{1 \cdot \pi}{2} = \frac{\pi}{6} + \frac{3\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$x = \frac{\pi}{6} + \frac{2 \cdot \pi}{2} = \frac{\pi}{6} + \frac{6\pi}{6} = \frac{7\pi}{6}$$

$$x = \frac{\pi}{6} + \frac{3 \cdot \pi}{2} = \frac{\pi}{6} + \frac{9\pi}{6} = \frac{10\pi}{6} = \frac{5\pi}{3}$$

The solutions that satisfy $0 \le x < 2\pi$ are $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, and $\frac{5\pi}{3}$.

2b. Solve the equation: $\sin\frac{x}{3} = \frac{1}{2},\ 0 \le x < 2\pi.$

The expression involving sine is already isolated on the left. In one period, the value of sine is $\frac{1}{2}$ at $\frac{\pi}{6}$ in quadrant I and at $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in quadrant II.

Thus,

$$\frac{x}{3} = \frac{\pi}{6} + 2n\pi \text{ or } \frac{x}{3} = \frac{5\pi}{6} + 2n\pi,$$

where n is any integer. Multiplying by 3 in each, we obtain

$$x = \frac{\pi}{2} + 6n\pi \text{ or } x = \frac{5\pi}{2} + 6n\pi,$$

where n is any integer. Letting $n = 0$ in $x = \frac{\pi}{2} + 6n\pi$, we obtain $x = \frac{\pi}{2}$. No other integer n results in a value of $x = \frac{\pi}{2} + 6n\pi$ that satisfies $0 \le x < 2\pi$. For $x = \frac{5\pi}{2} + 6n\pi$, there is no integer n that results in a value of x that satisfies $0 \le x < 2\pi$.

The only solution is $\frac{\pi}{2}$.

2b. Solve the equation: $\sin\frac{2\theta}{3} = -1,\ 0 \le \theta < 2\pi.$

Objective #3: Solve trigonometric equations quadratic in form.

✔ *Solved Problem #3*

3a. Solve the equation:
$2\sin^2 x - 3\sin x + 1 = 0,\ 0 \le x < 2\pi.$

Notice that if we replace $\sin x$ with u on the left side of the equation we obtain $2u^2 - 3u + 1,$ which factors as $(2u-1)(u-1).$ Thus, the left side of the equation factors as $(2\sin x - 1)(\sin x - 1).$ Thus, we can solve this equation by factoring and setting each factor equal to 0.

$$2\sin^2 x - 3\sin x + 1 = 0$$
$$(2\sin x - 1)(\sin x - 1) = 0$$
$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$2\sin x = 1 \qquad \sin x = 1$$
$$\sin x = \frac{1}{2}$$

The equation $\sin x = \frac{1}{2}$ has two solutions satisfying $0 \le x < 2\pi$: $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. The equation $\sin x = 1$ has one solution satisfying $0 \le x < 2\pi$: $\frac{\pi}{2}$. The solutions are $\frac{\pi}{6}$, $\frac{\pi}{2}$, and $\frac{5\pi}{6}$.

3b. Solve the equation: $4\cos^2 x - 3 = 0,\ 0 \le x < 2\pi.$

We solve the equation by isolating the squared expression and then using the square root property.

$$4\cos^2 x - 3 = 0$$
$$4\cos^2 x = 3$$
$$\cos^2 x = \frac{3}{4}$$
$$\cos x = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos x = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$$

✎ *Pencil Problem #3* ✐

3a. Solve the equation:
$2\sin^2 x - \sin x - 1 = 0,\ 0 \le x < 2\pi.$

3b. Solve the equation: $4\cos^2 x - 1 = 0,\ 0 \le x < 2\pi.$

The equation $\cos x = \frac{\sqrt{3}}{2}$ has two solutions satisfying $0 \le x < 2\pi$: $\frac{\pi}{6}$ and $\frac{11\pi}{6}$. The equation $\cos x = -\frac{\sqrt{3}}{2}$ has two solutions satisfying $0 \le x < 2\pi$: $\frac{5\pi}{6}$ and $\frac{7\pi}{6}$. The solutions are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Objective #4: Use factoring to separate different functions in trigonometric equations.

✔ *Solved Problem #4*

4. Solve the equation: $\sin x \tan x = \sin x,\ 0 \le x < 2\pi.$

We begin by subtracting $\sin x$ from both sides, obtaining 0 on the right. Then we factor out the common factor of $\sin x$ from the terms on the left and set each factor equal to 0.

$$\sin x \tan x = \sin x$$
$$\sin x \tan x - \sin x = 0$$
$$\sin x(\tan x - 1) = 0$$
$$\sin x = 0 \text{ or } \tan x - 1 = 0$$
$$\tan x = 1$$

The equation $\sin x = 0$ has solutions 0 and π satisfying $0 \le x < 2\pi$. The equation $\tan x = 1$ has solutions $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ satisfying $0 \le x < 2\pi$. The solutions are 0, $\frac{\pi}{4}$, π, and $\frac{5\pi}{4}$.

✎ *Pencil Problem #4* ✐

4. Solve the equation:
$\sin x + 2\sin x \cos x = 0,\ 0 \le x < 2\pi.$

Objective #5: Use identities to solve trigonometric equations.

✔ ***Solved Problem #5***

5a. Solve the equation: $\cos 2x + \sin x = 0,\ 0 \le x < 2\pi.$

We use the double-angle formula $\cos 2x = 1 - 2\sin^2 x$ to rewrite the left side in terms of sine only so the equation becomes quadratic in form.

$$\cos 2x + \sin x = 0$$
$$1 - 2\sin^2 x + \sin x = 0$$
$$-2\sin^2 x + \sin x + 1 = 0$$
$$2\sin^2 x - \sin x - 1 = 0$$
$$(2\sin x + 1)(\sin x - 1) = 0$$
$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
$$2\sin x = -1 \qquad \sin x = 1$$
$$\sin x = -\frac{1}{2}$$

The equation $\sin x = -\frac{1}{2}$ has solutions $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$ satisfying $0 \le x < 2\pi.$ The equation $\sin x = 1$ has only the solution $\frac{\pi}{2}$ satisfying $0 \le x < 2\pi.$ The solutions are $\frac{\pi}{2},\ \frac{7\pi}{6},$ and $\frac{11\pi}{6}.$

✎ ***Pencil Problem #5*** ✐

5a. Solve the equation: $\sin 2x = \cos x,\ 0 \le x < 2\pi.$

5b. Solve the equation: $\cos x - \sin x = -1,\ 0 \le x < 2\pi.$

We begin by squaring each side of the equation and then applying an identity. Remember that squaring may introduce extraneous solutions, so it is important to check all proposed solutions.

$$\cos x - \sin x = -1$$
$$(\cos x - \sin x)^2 = (-1)^2$$
$$\cos^2 x - 2\cos x \sin x + \sin^2 x = 1$$
$$(\cos^2 x + \sin^2 x) - 2\cos x \sin x = 1$$

Next we apply the identity $\sin^2 x + \cos^2 x = 1.$

$$1 - 2\cos x \sin x = 1$$
$$-2\cos x \sin x = 0$$
$$\cos x = 0 \text{ or } \sin x = 0$$

The first equation has solutions $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ satisfying $0 \le x < 2\pi.$ The second equation has solutions 0 and π satisfying $0 \le x < 2\pi.$

Check $\frac{\pi}{2}$: $\cos\frac{\pi}{2} - \sin\frac{\pi}{2} = -1?$
$$0 - 1 = -1?$$
$$-1 = -1, \text{ true}$$

Check $\frac{3\pi}{2}$: $\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = -1?$
$$0 - (-1) = -1?$$
$$1 = -1, \text{ false}$$

Check 0: $\cos 0 - \sin 0 = -1?$
$$1 - 0 = -1?$$
$$1 = -1, \text{ false}$$

Check π: $\cos\pi - \sin\pi = -1?$
$$-1 - 0 = -1?$$
$$-1 = -1, \text{ true}$$

The solutions are $\frac{\pi}{2}$ and $\pi.$

5b. Solve the equation: $\sin x + \cos x = 1,\ 0 \le x < 2\pi.$

Objective #6: Use a calculator to solve trigonometric equations.

✔ *Solved Problem #6*

6a. Solve $\tan x = 3.1044$, correct to four decimal places, for $0 \le x < 2\pi$.

Note that tangent is positive in quadrants I and III and that $\tan^{-1} 3.1044$ will be a value in quadrant I.

The solution in quadrant I is
$x = \tan^{-1} 3.1044 \approx 1.2592.$

The solution in quadrant III is
$x \approx \pi + 1.2592 \approx 4.4008.$

The solutions are 1.2592 and 4.4008.

6b. Solve $\sin x = -0.2315$, correct to four decimal places, for $0 \le x < 2\pi$.

Sine is negative in quadrants III and IV and $\sin^{-1} 0.2315 \approx 0.2336$ is in quadrant I.

The solution in quadrant III is
$x \approx \pi + 0.2336 \approx 3.3752.$

The solution in quadrant IV is
$x \approx 2\pi - 0.2336 \approx 6.0496.$

The solutions are 3.3752 and 6.0496.

✎ *Pencil Problem #6* ✐

6a. Solve $\sin x = 0.8246$, correct to four decimal places, for $0 \le x < 2\pi$.

6b. Solve $\tan x = -3$, correct to four decimal places, for $0 \le x < 2\pi$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. $\theta = \frac{\pi}{6} + 2n\pi,\ \theta = \frac{5\pi}{6} + 2n\pi,$ where n is any integer *(5.5 #21)*

2a. $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$ *(5.5 #29)* **2b.** no solution *(5.5 #33)*

3a. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ *(5.5 #39)* **3b.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ *(5.5 #47)*

4. $0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ *(5.5 #59)*

5a. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$ *(5.5 #69)* **5b.** $0, \frac{\pi}{2}$ *(5.5 #77)*

6a. 0.9695, 2.1721 *(5.5 #85)* **6b.** 1.8926, 5.0342 *(5.5 #89)*