

Section 3.1

Exponential Functions

Shop 'til You Drop!

Are you just browsing? Take your time.
 Researchers know, to the dollar, the average amount the typical consumer spends at the shopping mall. And the longer you stay, the more you spend. So if you say you're just browsing, that's just fine with the mall merchants. Browsing is time and, as we will explore in this section, time is money.

Objective #1: Evaluate exponential functions.

✓ Solved Problem #1

1. The exponential function $f(x) = 42.2(1.56)^x$ models the average amount spent, $f(x)$, in dollars, at a shopping mall after x hours. What is the average amount spent, to the nearest dollar, after three hours?

$$f(x) = 42.2(1.56)^x$$

$$f(3) = 42.2(1.56)^3$$

$$\approx 160.20876$$

$$\approx 160$$

The average amount spent after three hours at a mall is \$160.

✎ Pencil Problem #1 ✎

1. The exponential function $f(x) = 574(1.026)^x$ models the population of India, $f(x)$, in millions, x years after 1974. Find India's population, to the nearest million, in the year 2028.

Objective #2: Graph exponential functions.

✓ Solved Problem #2

- 2a. Graph: $f(x) = 3^x$

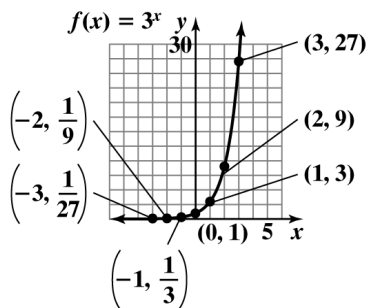
Make a table of values:

x	$f(x) = 3^x$	(x, y)
-3	$3^{-3} = \frac{1}{27}$	$\left(-3, \frac{1}{27}\right)$
-2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$3^0 = 1$	$(0, 1)$
1	$3^1 = 3$	$(1, 3)$
2	$3^2 = 9$	$(2, 9)$
3	$3^3 = 27$	$(3, 27)$

✎ Pencil Problem #2 ✎

- 2a. Graph: $f(x) = 4^x$

Plot the points in the table and connect with a smooth curve.

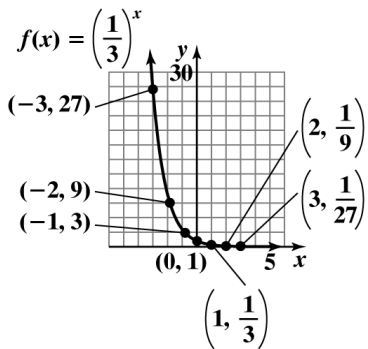


2b. Graph: $f(x) = \left(\frac{1}{3}\right)^x$

Make a table of values:

x	$f(x) = \left(\frac{1}{3}\right)^x$	(x, y)
-3	$\left(\frac{1}{3}\right)^{-3} = 27$	$(-3, 27)$
-2	$\left(\frac{1}{3}\right)^{-2} = 9$	$(-2, 9)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$	$(-1, 3)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

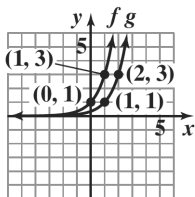
Plot the points in the table and connect with a smooth curve.



2b. Graph: $g(x) = \left(\frac{3}{2}\right)^x$

- 2c. Use the graph of $f(x) = 3^x$ to obtain the graph of $g(x) = 3^{x-1}$.

The graph of $g(x) = 3^{x-1}$ is the graph of $f(x) = 3^x$ shifted 1 unit to the right. We identified two points on the graph of f , which we graphed in Solved Problem 2a, and added 1 to each of the x -coordinates.



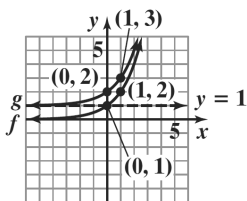
$$f(x) = 3^x$$

$$g(x) = 3^{x-1}$$

- 2c. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^{x+1}$.

- 2d. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x + 1$.

The graph of $g(x) = 2^x + 1$ is the graph of $f(x) = 2^x$ shifted up 1 unit. We identified two points on the graph of f and added 1 to each of the y -coordinates. The asymptote is also shifted up 1 unit.



$$f(x) = 2^x$$

$$g(x) = 2^x + 1$$

- 2d. Use the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x - 1$.

Objective #3: Evaluate functions with base e .

✓ **Solved Problem #3**

3. The exponential function $f(x) = 1066e^{0.042x}$ models the gray wolf population of the Western Great Lakes, $f(x)$, x years after 1978. If trends continue, project the gray wolf's population in the recovery area in 2012.

2012 is 34 years after 1978.

$$f(x) = 1066e^{0.042x}$$

$$f(34) = 1066e^{0.042(34)} \approx 4446$$

In 2012 the gray wolf population of the Western Great Lakes was projected to be about 4446.

✎ **Pencil Problem #3**

3. The exponential function $g(x) = 6875e^{0.077x}$ models the average cost of a family health insurance plan x years after 2000. What was the average cost of a family health insurance plan in 2011?

Objective #4: Use compound interest formulas.

✓ Solved Problem #4

4a. A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to quarterly compounding.

$$A = P \left(1 + \frac{r}{n} \right)^{nt}$$

$$= \$10,000 \left(1 + \frac{0.08}{4} \right)^{4 \cdot 5}$$

$$\approx \$14,859.47$$

4b. A sum of \$10,000 is invested at an annual rate of 8%. Find the balance in the account after 5 years subject to continuous compounding.

$$A = Pe^{rt}$$

$$= \$10,000e^{0.08(5)}$$

$$\approx \$14,918.25$$

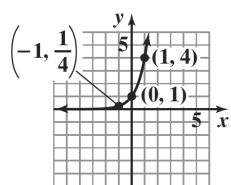
✎ Pencil Problem #4 ✎

4a. A sum of \$10,000 is invested at an annual rate of 5.5%. Find the balance in the account after 5 years subject to monthly compounding.

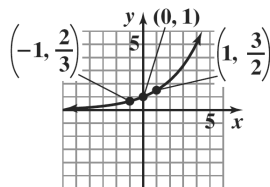
4b. A sum of \$10,000 is invested at an annual rate of 5.5%. Find the balance in the account after 5 years subject to continuous compounding.

Answers for Pencil Problems (Textbook Exercise references in parentheses):

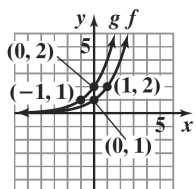
1. 2295 million (3.1 #65c)



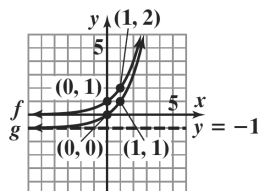
2a. $f(x) = 4^x$ (3.1 #11)



2b. $g(x) = \left(\frac{3}{2}\right)^x$ (3.1 #13)



2c. $f(x) = 2^x$
 $g(x) = 2^{x+1}$ (3.1 #25)



2d. $f(x) = 2^x$
 $g(x) = 2^x - 1$ (3.1 #27)

3. \$16,037 (3.1 #71b)

4a. \$13,157.04 (3.1 #53c) **4b.** \$13,165.31 (3.1 #53d)

Section 3.2

Logarithmic Functions

Speak Up!

The loudness level of a sound is measured in decibels. Decibel levels range from 0, a barely audible sound, to 160, a sound resulting in a ruptured eardrum.

We will see that decibels can be modeled by a logarithmic function, the topic of this section.

Objective #1: Change from logarithmic to exponential form.

Solved Problem #1

1. Write the equation $2 = \log_b 25$ in its equivalent exponential form.

$$2 = \log_b 25$$
$$b^2 = 25$$

Pencil Problem #1

1. Write the equation $\log_6 216 = y$ in its equivalent exponential form.

Objective #2: Change from exponential to logarithmic form.

Solved Problem #2

2. Write the equation $2^5 = x$ in its equivalent logarithmic form.

$$2^5 = x$$
$$5 = \log_2 x$$

Pencil Problem #2

2. Write the equation $7^y = 200$ in its equivalent logarithmic form.

Objective #3: Evaluate logarithms.

Solved Problem #3

- 3a. Evaluate: $\log_5 \frac{1}{125}$.

$$\log_5 \frac{1}{125} = -3 \text{ because}$$

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125}$$

Pencil Problem #3

3. Evaluate: $\log_5 \frac{1}{5}$.

3b. Evaluate: $\log_{36} 6$.

$$\log_{36} 6 = \frac{1}{2} \text{ because } 36^{\frac{1}{2}} = \sqrt{36} = 6.$$

3b. Evaluate: $\log_4 16$.

Objective #4: Use basic logarithmic properties.

✓ Solved Problem #4

4. Evaluate $\log_7 7^8$.

Because $\log_b b^x = x$, we conclude $\log_7 7^8 = 8$.

✎ Pencil Problem #4

4. Evaluate $\log_4 1$.

Objective #5: Graph logarithmic functions.

✓ Solved Problem #5

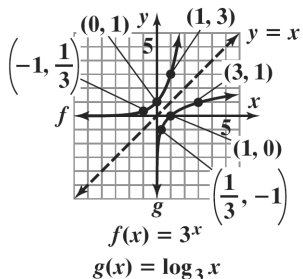
5. Graph $f(x) = 3^x$ and $g(x) = \log_3 x$ in the same rectangular coordinate system.

Set up a table of coordinates for $f(x) = 3^x$.

x	-2	-1	0	1	2	3
$f(x) = 3^x$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

Reverse these coordinates to obtain the coordinates of $g(x) = \log_3 x$.

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$g(x) = \log_3 x$	-2	-1	0	1	2	3



✎ Pencil Problem #5

5. Graph $f(x) = (\frac{1}{2})^x$ and $g(x) = \log_{\frac{1}{2}} x$ in the same rectangular coordinate system.

Objective #6: Find the domain of a logarithmic function. **Solved Problem #6**

6. Find the domain of
- $f(x) = \log_4(x-5)$
- .

$$x-5 > 0$$

$$x > 5$$

The domain of f is $(5, \infty)$. **Pencil Problem #6** 

6. Find the domain of
- $f(x) = \log(2-x)$
- .

Objective #7: Use common logarithms. **Solved Problem #7**

7. The percentage of adult height attained by a boy who is
- x
- years old can be modeled by
- $f(x) = 29 + 48.8\log(x+1)$
- , where
- x
- represents the boy's age and
- $f(x)$
- represents the percentage of his adult height. Approximately what percentage of his adult height has a boy attained at age ten?

$$f(x) = 29 + 48.8\log(x+1)$$

$$f(10) = 29 + 48.8\log(10+1)$$

$$= 29 + 48.8\log 11$$

$$\approx 80$$

A 10-year-old boy has attained approximately 80% of his adult height.

 **Pencil Problem #7** 

7. The percentage of adult height attained by a girl who is
- x
- years old can be modeled by
- $f(x) = 62 + 35\log(x-4)$
- , where
- x
- represents the girl's age and
- $f(x)$
- represents the percentage of her adult height. Approximately what percentage of her adult height has a girl attained at age 13?

Objective #8: Use natural logarithms. **Solved Problem #8**

- 8a. Find the domain of
- $f(x) = \ln(4-x)$
- .

The domain of f consists of all x for which $4-x > 0$.

$$4-x > 0$$

$$-x > -4$$

$$x < 4$$

The domain of f is $(-\infty, 4)$. **Pencil Problem #8** 

- 8a. Find the domain of
- $f(x) = \ln(x+2)$
- .

8b. The function $f(x) = 13.4 \ln x - 11.6$ models the temperature increase, $f(x)$, in an enclosed vehicle after x minutes when the outside air temperature is between 72°F and 96°F . Use the function to find the temperature increase, to the nearest degree, after 30 minutes.

Evaluate the function at 30.

$$f(x) = 13.4 \ln x - 11.6$$

$$f(30) = 13.4 \ln 30 - 11.6$$

$$\approx 34$$

The temperature will increase by about 34° after 30 minutes.

8b. The function $f(x) = -7.2 \ln x + 53$ models the percentage of first-year college men expressing antifeminist views x years after 1969. Use the function to project the percentage of first-year college men who will express antifeminist views in 2015. Round to one decimal place.

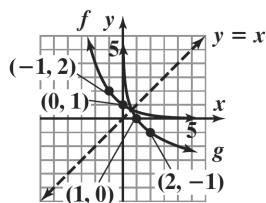
Answers for Pencil Problems (Textbook Exercise references in parentheses):

1. $6^y = 216$ (3.2 #7)

2. $\log_7 200 = y$ (3.2 #19)

3a. -1 (3.2 #21) 3b. 2 (3.2 #21)

4. 0 (3.2 #37)



$$f(x) = \left(\frac{1}{2}\right)^x$$

5. $g(x) = \log_{1/2} x$ (3.2 #45)

6. $(-\infty, 2)$ (3.2 #49)

7. 95.4% (3.2 #87)

8a. $(-2, \infty)$ (3.2 #81)

8b. 24.2% (3.2 #115)

Section 3.3

Properties of Logarithms

How Smart Is This Chimp?

Scientists are often amazed at what a chimpanzee can learn. These scientists are not simply interested in *what* a chimp can learn, but also in a particular chimp's *maximum capacity* for learning, and *how long* the learning takes.

A typical chimpanzee learning sign language can master a maximum of 65 signs. In the Exercise Set of the textbook, one application will take a look at the number of weeks it will take for a chimp to master 30 signs.

Objective #1: Use the product rule.

 **Solved Problem #1**

1. Use the product rule to expand: $\log(100x)$.

$$\begin{aligned}\log(100x) &= \log 100 + \log x \\ &= \log 10^2 + \log x \\ &= 2 + \log x\end{aligned}$$

 **Pencil Problem #1** 

1. Use the product rule to expand: $\log_5(7 \cdot 3)$.

Objective #2: Use the quotient rule.

 **Solved Problem #2**

2. Use the quotient rule to expand: $\ln\left(\frac{e^5}{11}\right)$.

$$\begin{aligned}\ln\left(\frac{e^5}{11}\right) &= \ln e^5 - \ln 11 \\ &= 5 - \ln 11\end{aligned}$$

 **Pencil Problem #2** 

2. Use the quotient rule to expand: $\log_4\left(\frac{64}{y}\right)$.

Objective #3: Use the power rule.

 **Solved Problem #3**

- 3a. Use the power rule to expand: $\log_6 3^9$

$$\log_6 3^9 = 9 \log_6 3$$

 **Pencil Problem #3** 

- 3a. Use the power rule to expand: $\log_b x^3$.

3b. Use the power rule to expand: $\log(x+4)^2$

$$\log(x+4)^2 = 2\log(x+4)$$

3b. Use the power rule to expand: $\log N^{-6}$

Objective #4: Expand logarithmic expressions.

 **Solved Problem #4**

4a. Use logarithmic properties to expand: $\ln \sqrt[3]{x}$.

$$\begin{aligned} \ln \sqrt[3]{x} &= \ln x^{\frac{1}{3}} \\ &= \frac{1}{3} \ln x \end{aligned}$$

 **Pencil Problem #4** 

4a. Use logarithmic properties to expand: $\ln \sqrt[5]{x}$.

4b. Use logarithmic properties to expand:
 $\log_b(x^4 \sqrt[3]{y})$.

$$\begin{aligned} \log_b(x^4 \sqrt[3]{y}) &= \log_b x^4 + \log_b \sqrt[3]{y} \\ &= \log_b x^4 + \log_b y^{\frac{1}{3}} \\ &= 4\log_b x + \frac{1}{3}\log_b y \end{aligned}$$

4b. Use logarithmic properties to expand: $\log_b(x^2 y)$.

4c. Use logarithmic properties to expand: $\log_5\left(\frac{\sqrt{x}}{25y^3}\right)$.

$$\begin{aligned} \log_5\left(\frac{\sqrt{x}}{25y^3}\right) &= \log_5\left(\frac{x^{\frac{1}{2}}}{25y^3}\right) \\ &= \log_5 x^{\frac{1}{2}} - \log_5(25y^3) \\ &= \log_5 x^{\frac{1}{2}} - (\log_5 25 + \log_5 y^3) \\ &= \log_5 x^{\frac{1}{2}} - \log_5 25 - \log_5 y^3 \\ &= \frac{1}{2}\log_5 x - 2 - 3\log_5 y \end{aligned}$$

4c. Use logarithmic properties to expand: $\log_6\left(\frac{36}{\sqrt{x+1}}\right)$.

Objective #5: Condense logarithmic expressions.
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Solved Problem #5

5a. Write as a single logarithm: $\log 25 + \log 4$.

$$\begin{aligned}\log 25 + \log 4 &= \log(25 \cdot 4) \\ &= \log 100 \\ &= 2\end{aligned}$$

Pencil Problem #5

5a. Write as a single logarithm: $\log 5 + \log 2$.

5b. Write as a single logarithm: $2 \ln x + \frac{1}{3} \ln(x+5)$.

$$\begin{aligned}2 \ln x + \frac{1}{3} \ln(x+5) &= \ln x^2 + \ln(x+5)^{\frac{1}{3}} \\ &= \ln x^2 + \ln \sqrt[3]{x+5} \\ &= \ln(x^2 \sqrt[3]{x+5})\end{aligned}$$

5b. Write as a single logarithm: $4 \ln(x+6) - 3 \ln x$.

5c. Write as a single logarithm:
 $\frac{1}{4} \log_b x - 2 \log_b 5 - 10 \log_b y$.

$$\begin{aligned}\frac{1}{4} \log_b x - 2 \log_b 5 - 10 \log_b y &= \log_b x^{\frac{1}{4}} - \log_b 5^2 - \log_b y^{10} \\ &= \log_b x^{\frac{1}{4}} - (\log_b 5^2 + \log_b y^{10}) \\ &= \log_b \sqrt[4]{x} - (\log_b 25 + \log_b y^{10}) \\ &= \log_b \sqrt[4]{x} - \log_b 25y^{10} \\ &= \log_b \left(\frac{\sqrt[4]{x}}{25y^{10}} \right)\end{aligned}$$

5c. Write as a single logarithm:
 $3 \ln x + 5 \ln y - 6 \ln z$.

Objective #6: Use the change-of-base property.**✓ Solved Problem #6****6a.** Use common logarithms to evaluate $\log_7 2506$.

$$\log_7 2506 = \frac{\log 2506}{\log 7}$$

$$\approx 4.02$$

6b. Use natural logarithms to evaluate $\log_7 2506$.

$$\log_7 2506 = \frac{\ln 2506}{\ln 7}$$

$$\approx 4.02$$

 Pencil Problem #6 **6a.** Use common logarithms to evaluate $\log_{0.1} 17$.**6b.** Use natural logarithms to evaluate $\log_{0.1} 17$.**Answers for Pencil Problems (Textbook Exercise references in parentheses):**

1. $\log_5 7 + \log_5 3$ (3.3 #1)

2. $3 - \log_4 y$ (3.3 #11)

3a. $3 \log_b x$ (3.3 #15)

3b. $-6 \log N$ (3.3 #17)

4a. $\frac{1}{5} \ln x$ (3.3 #19) **4b.** $2 \log_b x + \log_b y$ (3.3 #21) **4c.** $2 - \frac{1}{2} \log_6(x+1)$ (3.3 #25)

5a. 1 (3.3 #41) **5b.** $\ln \left[\frac{(x+6)^4}{x^3} \right]$ (3.3 #59) **5c.** $\ln \left(\frac{x^3 y^5}{z^6} \right)$ (3.3 #61)

6a. -1.2304 (3.3 #75)

6b. -1.2304 (3.3 #75)

Section 3.4

Exponential and Logarithmic Equations

Under the Sea !

Though it can be pitch black in the depths of the ocean, sunlight is visible as you get closer to the surface. About 12% of the surface sunlight reaches a depth of 20 feet and about 1.6% reaches to a depth of 100 feet.

In the applications of this section, you will use an exponential function to determine the depths that correspond to various percentages of light.

Objective #1: Use like bases to solve exponential equations.

 **Solved Problem #1**

1a. Solve: $5^{3x-6} = 125$.

$$5^{3x-6} = 125$$

$$5^{3x-6} = 5^3$$

$$3x - 6 = 3$$

$$3x = 9$$

$$x = 3$$

The solution set is $\{3\}$.

1b. Solve: $8^{x+2} = 4^{x-3}$.

$$8^{x+2} = 4^{x-3}$$

$$(2^3)^{x+2} = (2^2)^{x-3}$$

$$2^{3(x+2)} = 2^{2(x-3)}$$

$$3(x+2) = 2(x-3)$$

$$3x + 6 = 2x - 6$$

$$x + 6 = -6$$

$$x = -12$$

The solution set is $\{-12\}$.

 **Pencil Problem #1** 

1a. Solve: $4^{2x-1} = 64$.

1b. Solve: $8^{x+3} = 16^{x-1}$.

Objective #2: Use logarithms to solve exponential equations.

 **Solved Problem #2**

2a. Solve: $10^x = 8000$.

Take the common log of both sides of the equation.

$$10^x = 8000$$

$$\log 10^x = \log 8000$$

$$x = \log 8000$$

$$x \approx 3.90$$

The solution set is $\{\log 8000 \approx 3.90\}$.

 **Pencil Problem #2** 

2a. Solve: $10^x = 3.91$.

2b. Solve: $7e^{2x} - 5 = 58$.

Isolate the exponential expression, then take the natural log of both sides of the equation.

$$7e^{2x} - 5 = 58$$

$$7e^{2x} = 63$$

$$e^{2x} = 9$$

$$\ln e^{2x} = \ln 9$$

$$2x = \ln 9$$

$$x = \frac{\ln 9}{2}$$

$$x = \frac{\ln 3^2}{2}$$

$$x = \frac{2 \ln 3}{2}$$

$$x = \ln 3$$

$$x \approx 1.10$$

The solution set is $\{\ln 3 \approx 1.10\}$.

2b. Solve: $7^{x+2} = 410$.

Objective #3: Use exponential form to solve logarithmic equations.

Solved Problem #3

3a. Solve: $4\ln(3x) = 8$.

$$4\ln(3x) = 8$$

$$\ln(3x) = 2$$

$$e^2 = 3x$$

$$\frac{e^2}{3} = x$$

Check:

$$4\ln(3x) = 8$$

$$4\ln\left[3\left(\frac{e^2}{3}\right)\right] = 8$$

$$4\ln(e^2) = 8$$

$$4 \cdot 2 = 8$$

$$8 = 8, \text{ true}$$

The solution set is $\left\{\frac{e^2}{3}\right\}$.

Pencil Problem #3

3a. Solve: $\log_4(x+5) = 3$.

3b. Solve: $\log x + \log(x-3) = 1$.

$$\log x + \log(x-3) = 1$$

$$\log(x^2 - 3x) = 1$$

$$10^1 = x^2 - 3x$$

$$0 = x^2 - 3x - 10$$

$$0 = (x+2)(x-5)$$

$$x+2=0 \quad \text{or} \quad x-5=0$$

$$x = -2$$

$$x = 5$$

Check -2:

$$\log x + \log(x-3) = 1$$

$$\log(-2) + \log(-2-3) = 1$$

-2 does not check.

Check 5:

$$\log x + \log(x-3) = 1$$

$$\log 5 + \log(5-3) = 1$$

$$\log 5 + \log 2 = 1$$

$$\log 10 = 1$$

$$1 = 1, \text{ true}$$

The solution set is $\{5\}$.

3b. Solve: $\log_5 x + \log_5(4x-1) = 1$.

Objective #4: Use the one-to-one property of logarithms to solve logarithmic equations.

 **Solved Problem #4**

4. Solve: $\ln(x-3) = \ln(7x-23) - \ln(x+1)$

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(x-3) = \ln\left(\frac{7x-23}{x+1}\right)$$

$$x-3 = \frac{7x-23}{x+1}$$

$$(x+1)(x-3) = (x+1)\frac{7x-23}{x+1}$$

$$x^2 - 2x - 3 = 7x - 23$$

$$x^2 - 9x + 20 = 0$$

$$(x-4)(x-5) = 0$$

$$x-4 = 0 \quad \text{or} \quad x-5 = 0$$

$$x = 4 \quad \quad \quad x = 5$$

Check 4:

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(4-3) = \ln(7 \cdot 4 - 23) - \ln(4+1)$$

$$\ln 1 = \ln 5 - \ln 5$$

$$0 = 0, \text{ true}$$

Check 5:

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

$$\ln(5-3) = \ln(7 \cdot 5 - 23) - \ln(5+1)$$

$$\ln 2 = \ln 12 - \ln 6$$

$$\ln 2 = \ln\left(\frac{12}{6}\right)$$

$$\ln 2 = \ln 2, \text{ true}$$

The solution set is $\{4, 5\}$.

 **Pencil Problem #4** 

4. Solve: $\log(x+4) - \log 2 = \log(5x+1)$

Objective #5: Solve applied problems involving exponential and logarithmic equations.

 **Solved Problem #5**

- 5a.** Medical research indicates that the risk of having a car accident increases exponentially as the concentration of alcohol in the blood increases. The risk is modeled by $R = 6e^{12.77x}$ where x is the blood alcohol concentration and R , given as a percent, is the risk of having a car accident. What blood alcohol concentration corresponds to a 7% risk of a car accident?

$$\begin{aligned}
 R &= 6e^{12.77x} \\
 6e^{12.77x} &= 7 \\
 e^{12.77x} &= \frac{7}{6} \\
 \ln e^{12.77x} &= \ln \frac{7}{6} \\
 12.77x &= \ln \frac{7}{6} \\
 x &= \frac{\ln \frac{7}{6}}{12.77} \\
 x &\approx 0.01
 \end{aligned}$$

For a blood alcohol concentration of 0.01, the risk of a car accident is 7%.

 **Pencil Problem #5**

- 5a.** The formula $A = 37.3e^{0.0095t}$ models the population of California, A , in millions, t years after 2010. When will the population of California reach 40 million?

- 5b.** How long, to the nearest tenth of a year, will it take \$1000 to grow to \$3600 at 8% annual interest compounded quarterly?

$$\begin{aligned}
 A &= P \left(1 + \frac{r}{n} \right)^{nt} \\
 3600 &= 1000 \left(1 + \frac{0.08}{4} \right)^{4t} \\
 3.6 &= 1.02^{4t} \\
 1.02^{4t} &= 3.6 \\
 \ln 1.02^{4t} &= \ln 3.6 \\
 4t \ln 1.02 &= \ln 3.6 \\
 t &= \frac{\ln 3.6}{4 \ln 1.02} \\
 t &\approx 16.2
 \end{aligned}$$

After approximately 16.2 years, the \$1000 will grow to \$3600.

- 5b.** How long, to the nearest tenth of a year, will it take \$8000 to grow to \$16,000 at 8% annual interest compounded continuously?

Answers for Pencil Problems (*Textbook Exercise references in parentheses*):

1a. $\{a\}$ (3.4 #7) **1b.** $\{13\}$ (3.4 #19)

2a. $\{\log 3.91 \approx 0.59\}$ (3.4 #23)

2b. $\left\{\frac{\ln 410}{\ln 7} - 2 \approx 1.09\right\}$ (3.4 #37)

3a. $\{59\}$ (3.4 #53)

3b. $\left\{\frac{5}{4}\right\}$; note: reject -1 (3.4 #67)

4. $\left\{\frac{2}{9}\right\}$ (3.4 #83)

5a. 2017 (3.4 #103b)

5b. 8.7 years (3.4 #111)

Section 3.5

Exponential Growth and Decay; Modeling Data

NOT Tooth Decay!

One of algebra's many applications is to predict the behavior of variables. This can be done with *exponential growth* and *decay models*. With exponential growth or decay, quantities grow or decay at a rate directly proportional to their size. Populations that are growing exponentially grow extremely rapidly as they get larger because there are more adults to have offspring.

In this section we explore how to create such functions and how to use them to make predictions.

Objective #1: Model exponential growth and decay.

✓ Solved Problem #1

- In 2000, the population of Africa was 807 million and by 2011 it had grown to 1052 million.
 - Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find the exponential growth function that models the data.

2011 is 11 years after 2000.
Thus, when $t = 11$, $A = 1052$.
 $A_0 = 807$

Substitute these values to find k .

$$A = A_0e^{kt}$$
$$1052 = 807e^{k(11)}$$

Solve for k .

$$1052 = 807e^{k(11)}$$
$$\frac{1052}{807} = e^{11k}$$
$$\ln \frac{1052}{807} = \ln e^{11k}$$
$$k = \frac{\ln \frac{1052}{807}}{11}$$
$$k \approx 0.024$$

Thus, the growth function is $A = 807e^{0.024t}$.

✎ Pencil Problem #1 ✎

- In 2000, the population of Israel was approximately 6.04 million and by 2050 it is projected to grow to 10 million.
 - Use the exponential growth model $A = A_0e^{kt}$, in which t is the number of years after 2000, to find an exponential growth function that models the data.

1b. By which year will Africa's population reach 2000 million, or two billion?

$$A = 807e^{0.024t}$$

$$2000 = 807e^{0.024t}$$

$$\frac{2000}{807} = e^{0.024t}$$

$$\ln \frac{2000}{807} = \ln e^{0.024t}$$

$$\ln \frac{2000}{807} = 0.024t$$

$$t = \frac{\ln \frac{2000}{807}}{0.024} \approx 38$$

Africa's population will reach 2000 million approximately 38 years after 2000, or 2038.

1b. In which year will Israel's population be 9 million?

Objective #2: Use logistic growth models.

 **Solved Problem #2**

2. In a learning theory project, psychologists discovered that

$$f(t) = \frac{0.8}{1 + e^{-0.2t}}$$

is a model for describing the proportion of correct responses, $f(t)$, after t learning trials.

2a. Find the proportion of correct responses after 10 learning trials.

We substitute 10 for t in the logistic growth function.

$$f(10) = \frac{0.8}{1 + e^{-0.2(10)}} \approx 0.7$$

The proportion of correct responses after 10 trials is approximately 0.7.

 **Pencil Problem #2** 

2. The logistic growth function

$$f(t) = \frac{100,000}{1 + 5000e^{-t}}$$

describes the number of people, $f(t)$, who have become ill with influenza t weeks after its initial outbreak in a particular community.

2a. How many people were ill by the end of the fourth week?

- 2b.** What is the limiting size of $f(t)$, the proportion of correct responses, as continued learning trials take place?

The number in the numerator of the logistic growth function, 0.8, is the limiting size of the proportion of correct responses.

- 2b.** What is the limiting size of the population that becomes ill?

Objective #3: Use Newton's Law of Cooling.

 **Solved Problem #3**

- 3.** An object is heated to 100°C . It is left to cool in a room that has a temperature of 30°C . After 5 minutes, the temperature of the object is 80°C .
- 3a.** Use Newton's Law of Cooling, $T = C + (T_0 - C)e^{kt}$, to find a model for the temperature of the object, T , after t minutes.

The initial temperature is 100°C : $T_0 = 100$.

The constant temperature of the room is 30°C : $C = 30$.

$$\begin{aligned} T &= C + (T_0 - C)e^{kt} \\ &= 30 + (100 - 30)e^{kt} \\ &= 30 + 70e^{kt} \end{aligned}$$

After 5 minutes, the temperature of the object is 80°C : $T = 80$ when $t = 5$. Substitute these values into

$T = 30 + 70e^{kt}$ and solve for k . Begin by isolating the exponential expression on one side.

$$\begin{aligned} T &= 30 + 70e^{kt} \\ 80 &= 30 + 70e^{k(5)} \\ 50 &= 70e^{5k} \\ \frac{50}{70} &= e^{5k} \\ \frac{5}{7} &= e^{5k} \end{aligned}$$

(continued on next page)

 **Pencil Problem #3** 

- 3.** A bottle of juice initially has a temperature of 70°F . It is left to cool in a refrigerator that has a temperature of 45°F . After 10 minutes, the temperature of the juice is 55°F .
- 3a.** Use Newton's Law of Cooling, $T = C + (T_0 - C)e^{kt}$, to find a model for the temperature of the juice, T , after t minutes.

Now reverse sides, take the natural logarithm of each side and continue to solve for k .

$$e^{5k} = \frac{5}{7}$$

$$\ln e^{5k} = \ln\left(\frac{5}{7}\right)$$

$$5k = \ln\left(\frac{5}{7}\right)$$

$$k = \frac{\ln\left(\frac{5}{7}\right)}{5} \approx -0.0673$$

Substitute -0.0673 for k in $T = 30 + 70e^{kt}$. The model is $T = 30 + 70e^{-0.0673t}$.

3b. What is the temperature of the object after 20 minutes?

Substitute 20 for t in the model found in Solved Problem #3a and evaluate with a calculator.

$$T = 30 + 70e^{-0.0673(20)} \approx 48$$

After 20 minutes, the temperature of the object will be approximately 48°C .

3c. When will the temperature of the object be 35°C ?

Substitute 35 for T in the model found in Solved Problem #3a and solve for t .

$$35 = 30 + 70e^{-0.0673t}$$

$$5 = 70e^{-0.0673t}$$

$$\frac{5}{70} = e^{-0.0673t}$$

$$e^{-0.0673t} = \frac{1}{14}$$

$$\ln e^{-0.0673t} = \ln\left(\frac{1}{14}\right)$$

$$-0.0673t = \ln\left(\frac{1}{14}\right)$$

$$t = \frac{\ln\left(\frac{1}{14}\right)}{-0.0673} \approx 39$$

The object will be 35°C after approximately 39 minutes.

3b. What is the temperature of the juice after 15 minutes?

3c. When will the temperature of the juice be 50°F ?

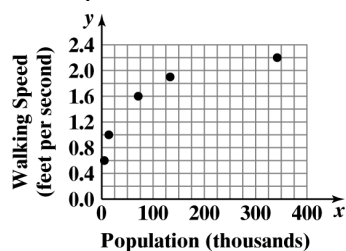
Objective #4: Choose an appropriate model for data.

✓ Solved Problem #4

4. The table shows the populations of various cities and the average walking speed of a person living in the city. Create a scatter plot for the data. Based on the scatter plot, what type of function would be a good choice for modeling the data?

Population (thousands)	Walking Speed (feet per second)
5.5	0.6
14	1.0
71	1.6
138	1.9
342	2.2

Scatter plot:



A logarithmic function would be a good choice for modeling the data.

✎ Pencil Problem #4 ✎

4. The table shows the percentage of miscarriages by women of various ages. Create a scatter plot for the data. Determine the type of function that would be a good choice for modeling the data?

Woman's Age	Percent of Miscarriages
22	9%
27	10%
32	13%
37	20%
42	38%
47	52%

Objective #5: Express an exponential model in base e .

✓ Solved Problem #5

5. Rewrite $y = 4(7.8)^x$ in terms of base e . Express the answer in terms of a natural logarithm and then round to three decimal places.

$$y = 4(7.8)^x$$

$$= 4e^{(\ln 7.8)x}$$

Rounded to three decimal places:

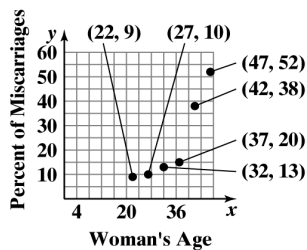
$$y = 4e^{2.054x}$$

✎ Pencil Problem #5

5. Rewrite $y = 100(4.6)^x$ in terms of base e . Express the answer in terms of a natural logarithm and then round to three decimal places.

Answers for Pencil Problems (Textbook Exercise references in parentheses):

- 1a. $A = 6.04e^{0.01t}$ (3.5 #7a)
- 1b. 2040 (3.5 #7b)
- 2a. approximately 1080 people (3.5 #37b)
- 2b. 100,000 people (3.5 #37c)
- 3a. $T = 45 + 25e^{-0.0916t}$ (3.5 #47a) 3b. 51°F (3.5 #47b) 3c. 18 minutes (3.5 #47c)



4. exponential function (3.5 #51)

5. $y = 100e^{(\ln 4.6)x} = 100e^{1.526x}$ (3.5 #57)