Section 11.1
Finding Limits Using Tables and Graphs

Knowing Your Limits

In order to succeed in calculus, you will need to understand *limits*. In this section, we take an intuitive approach to limits to help you understand what a limit is. You will learn the formal definition of a limit when you take calculus.

Many fascinating properties of the real numbers make calculus possible. In this section, as you consider *approaching a number*, remember that no matter how close two numbers may be to each other on a number line, there are always infinitely many real numbers between those two numbers.

So if you pick a number close to 0, but not equal to 0, I can pick a number even closer than yours. Then you can pick a new number closer to 0 than my number, and we could go back and forth forever, picking numbers closer to 0.

Objective #1: Understand limit notation.

✔ *Solved Problem #1*

1. Fill in the blank: The limit notation $\lim_{x\to 5} f(x) = 3$ means that the output values of the function *f* are approaching ______ as the input values are approaching ______.

$\lim_{x\to 5} f(x) = 3$ means that the output values of the function *f* are approaching 3 as the input values are approaching 5.

✎ *Pencil Problem #1* ✎

1. Fill in the blank: The limit notation $\lim_{x\to -4} g(x) = 7$ means that the output values of the function *g* are approaching ______ as the input values are approaching ______.

Objective #2: Find limits using tables.

✔ *Solved Problem #2*

2a. Construct a table to find $\lim_{x\to 3} 4x^2$.

As *x* gets closer to 3, but not equal to 3, we must determine the value that $4x^2$ is getting closer to. We choose several values of *x* that are less than 3, such as 2.99, 2.999, and 2.9999, and several values of *x* that are greater than 3, such as 3.01, 3.001, and 3.001, and evaluate $4x^2$ at each of these values.

(*continued on next page)*

✎ *Pencil Problem #2* ✎

2a. Construct a table to find $\lim_{x\to 2} 5x^2$.

The tables show the values we chose for x and the corresponding values of $4x^2$.

x approaches 3 from the left. $\rightarrow$

x	2.99	2.999	2.9999	$\rightarrow$
$4x^2$	35.7604	35.976004	35.99760004	$\rightarrow$

$\leftarrow$ x approaches 3 from the right.

x	$\leftarrow$	3.0001	3.001	3.01
$4x^2$	$\leftarrow$	36.00240004	36.024004	36.2404

It appears that as x gets closer to 3, the values of $4x^2$ get closer to 36.

We infer that $\lim_{x\to 3} 4x^2 = 36.$

2b. Construct a table to find $\lim_{x\to 0} \frac{\cos x - 1}{x}.$

We choose several values of x that are less than 0 and several values of x that are greater than 0 and then evaluate $\frac{\cos x - 1}{x}$ at each of these values. The values of x are in radians. We round the values of $\frac{\cos x - 1}{x}$ to five decimal places.

x approaches 0 from the left. $\rightarrow$

x	–0.01	–0.001	–0.0001	$\rightarrow$
$\frac{\cos x - 1}{x}$	0.00500	0.00050	0.00005	$\rightarrow$

$\leftarrow$ x approaches 0 from the right.

x	$\leftarrow$	0.0001	0.001	0.01
$\frac{\cos x - 1}{x}$	$\leftarrow$	–0.00005	–0.00050	–0.00500

It appears that as x gets closer to 0, the values of $\frac{\cos x - 1}{x}$ also get closer to 0.

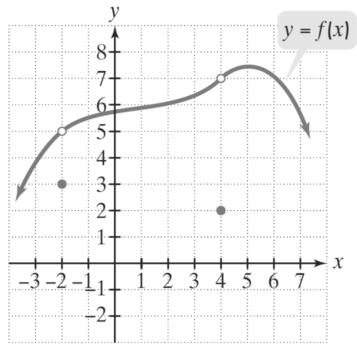
We infer that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$

2b. Construct a table to find $\lim_{x\to 0} \frac{\tan x}{x}.$

Objective #3: Find limits using graphs.

✔ *Solved Problem #3*

3a. Use the graph to find $\lim_{x\to -2} f(x)$ and $f(-2)$.



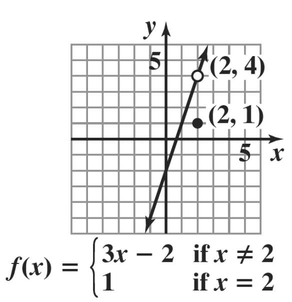
To find $\lim_{x\to -2} f(x)$, examine the graph of f near $x = -2$. As x gets closer to -2, the values of $f(x)$ get closer to the y-coordinate of the open dot on the left. The y-coordinate of this point is 5. We conclude that $\lim_{x\to -2} f(x) = 5$.

To find $f(-2)$, examine the graph at $x = -2$. The graph of f is shown by the closed dot with coordinates $(-2, 3)$. Thus, $f(-2) = 3$.

3b. Graph the function $f(x) = \begin{cases} 3x-2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

Use the graph to find $\lim_{x\to 2} f(x)$.

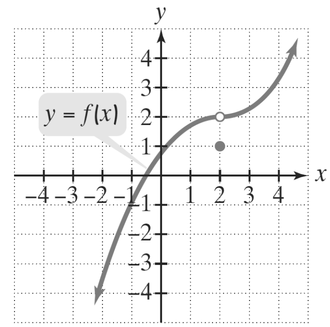
This is a piecewise function. To graph the piece defined by $f(x) = 3x - 2$, we use the slope, 3, and the y-intercept, -2. Since this piece does not include $x = 2$, we include a hole in the graph at $(2, 4)$. To graph the other piece, we plot the point $(2, 1)$.



To find $\lim_{x\to 2} f(x)$, examine the graph of f near $x = 2$. As x gets closer to 2, the values of $f(x)$ get closer to the y-coordinate of the open dot. The y-coordinate of this point is 4. We conclude that $\lim_{x\to 2} f(x) = 4$.

✎ *Pencil Problem #3* ✎

3a. Use the graph to find $\lim_{x\to 2} f(x)$ and $f(2)$.

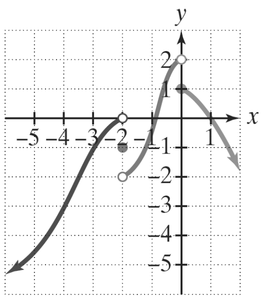


3b. Graph the function $f(x) = \begin{cases} x+1 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2. \end{cases}$ Use the graph to find $\lim_{x\to 2} f(x)$.

Objective #4: Find one-sided limits and use them to determine if a limit exists.

✔ *Solved Problem #4*

4. Use the graph to find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$, and $f(0)$ or state that the limit or function value does not exist.



To find $\lim_{x\to 0^-} f(x)$, examine the portion of the graph near, but to the left of $x = 0$. As x approaches 0 from the left, the values of $f(x)$ get closer to the y-coordinate of the open dot at (0, 2). Thus, $\lim_{x\to 0^-} f(x) = 2$.

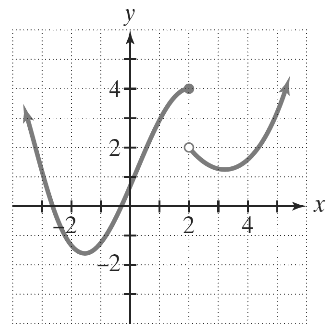
To find $\lim_{x\to 0^+} f(x)$, examine the portion of the graph near, but to the right of $x = 0$. As x approaches 0 from the right, the values of $f(x)$ get closer to the y-coordinate of the closed dot at (0, 1). Thus, $\lim_{x\to 0^+} f(x) = 1$.

Because $\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = 1$ are not equal, $\lim_{x\to 0} f(x)$ does not exist.

To find $f(0)$, examine the graph for $x = 0$. The closed dot at (0, 1) indicates that $f(0) = 1$.

✎ *Pencil Problem #4* ✐

4. Use the graph to find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$, and $f(2)$ or state that the limit or function value does not exist.



Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. 7; –4
2a. 20 *(11.1 #5)* **2b.** 1 *(11.1 #15)*
3a. $\lim_{x\to 2} f(x) = 2$; $f(2) = 1$ *(11.1 #21)* **3b.** 3 *(11.1 #47)*
4. $\lim_{x\to 2^-} f(x) = 4$; $\lim_{x\to 2^+} f(x) = 2$; $\lim_{x\to 2} f(x)$ does not exist; $f(2) = 4$ *(11.1 #27)*

Section 11.2
Finding Limits Using Properties of Limits

Establishing Your Limits

In the previous section, we explored limits using tables and graphs. While these methods help us to understand what limits are, they are not practical for determining the values of limits in most cases. Not only is it time consuming to construct a table or graph but it may also be difficult to find exact values for limits using these methods. If the value of a limit were $\frac{\pi}{2} \approx 1.570796...,$ how would you be able to figure that out from a table or graph?

In this section, we look at more efficient methods of determining limits. In the Exercise Set, you will see some amazing results when limits are applied to traveling at velocities approaching the speed of light.

Objective #1: Find limits of constant functions and the identity function.

✔ *Solved Problem #1*

1a. Find $\lim_{x\to 8} 11$.

Since the expression following the limit is a constant, the value of the limit is that constant: $\lim_{x\to a} c = c$.

$\lim_{x\to 8} 11 = 11$

1b. Find $\lim_{x\to 0}(-9)$.

Since the expression following the limit is a constant, the value of the limit is that constant: $\lim_{x\to a} c = c$.

$\lim_{x\to 0}(-9) = -9$

1c. Find $\lim_{x\to 19} x$.

Since the expression following the limit is the identity function, x, the value of the limit is the same as the number x is approaching: $\lim_{x\to a} x = a$.

$\lim_{x\to 19} x = 19$

✎ *Pencil Problem #1* ✐

1a. Find $\lim_{x\to 2} 8$.

1b. Find $\lim_{x\to 3}(-6)$.

1c. Find $\lim_{x\to 2} x$.

Objective #2: Find limits using properties of limits.

✔ *Solved Problem #2*

2a. Find $\lim_{x\to -5}(3x-7)$.

Use the rule for the limit of a difference: The limit of a difference is the difference of the limits. Then use the rule for the limit of a product: The limit of a product is a product of the limits.

$$\begin{aligned}\lim_{x\to -5}(3x-7) &= \lim_{x\to -5}(3x) - \lim_{x\to -5} 7\\ &= \lim_{x\to -5} 3\cdot \lim_{x\to -5} x - \lim_{x\to -5} 7\\ &= 3(-5)-7\\ &= -22\end{aligned}$$

2b. Find $\lim_{x\to 2}(-7x^3)$.

Use the rule for the limit of a monomial: The limit of a monomial as $x \to a$ is the monomial evaluated at a.

$$\lim_{x\to 2}(-7x^3) = -7(2)^3 = -7\cdot 8 = -56$$

2c. Find $\lim_{x\to 2}(7x^3+3x^2-5x+3)$.

Use the rule for the limit of a polynomial: The limit of a polynomial as $x \to a$ is the polynomial evaluated at a.

$$\begin{aligned}\lim_{x\to 2}(7x^3+3x^2-5x+3) &= 7\cdot 2^3 + 3\cdot 2^2 - 5\cdot 2 + 3\\ &= 7\cdot 8 + 3\cdot 4 - 5\cdot 2 + 3\\ &= 56+12-10+3\\ &= 61\end{aligned}$$

✎ *Pencil Problem #2* ✎

2a. Find $\lim_{x\to 6}(3x-4)$.

2b. Find $\lim_{x\to -2} 7x^2$.

2c. Find $\lim_{x\to 5}(x^2-3x-4)$.

2d. Find $\lim_{x\to 4}(3x-5)^3$.

Use the rule for the limit of a power: The limit of a power is the power of the limit. Then use the rule for the limit of a polynomial.

$$\lim_{x\to 4}(3x-5)^3 = [\lim_{x\to 4}(3x-5)]^3 = [3\cdot 4-5]^3 = 7^3 = 343$$

2d. Find $\lim_{x\to 2}(5x-8)^3$.

2e. Find $\lim_{x\to -1}\sqrt{6x^2-4}$.

Use the rule for the limit of a root: The limit of a root is the root of the limit. Then use the rule for the limit of a polynomial.

$$\lim_{x\to -1}\sqrt{6x^2-4} = \sqrt{\lim_{x\to -1}(6x^2-4)} = \sqrt{6(-1)^2-4} = \sqrt{2}$$

2e. Find $\lim_{x\to -4}\sqrt{x^2+9}$.

2f. Find $\lim_{x\to 2}\dfrac{x^2-4x+1}{3x-5}$.

First find the limit of the denominator as $x \to 2$.
$\lim_{x\to 2}(3x-5) = 3\cdot 2-5 = 1$

Since the limit of the denominator is not 0, we can use the rule for the limit of a quotient: The limit of a quotient is the quotient of the limits. Then use the rule for the limit of a polynomial in the numerator and denominator.

$$\lim_{x\to 2}\frac{x^2-4x+1}{3x-5} = \frac{\lim_{x\to 2}(x^2-4x+1)}{\lim_{x\to 2}(3x-5)} = \frac{2^2-4\cdot 2+1}{3\cdot 2-5} = \frac{-3}{1} = -3$$

2f. Find $\lim_{x\to 2}\dfrac{x^2-1}{x-1}$.

Objective #3: Find one-sided limits using properties of limits.

✔ Solved Problem #3

3. Consider the piecewise function

$$f(x)=\begin{cases}-1 & \text{if } x<1\\ \sqrt[3]{2x-1} & \text{if } x\geq 1.\end{cases}$$

Find $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, and $\lim_{x\to 1} f(x)$, or state that the limit does not exist.

To find $\lim_{x\to 1^-} f(x)$, we look at values of $f(x)$ when x is close to 1 but less than 1. Because x is less than 1, we use the first line of the function: $f(x)=-1$.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(-1)=-1$$

To find $\lim_{x\to 1^+} f(x)$, we look at values of $f(x)$ when x is close to 1 but greater than 1. Because x is greater than 1, we use the second line of the function: $f(x)=\sqrt[3]{2x-1}$.

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt[3]{2x-1}=\sqrt[3]{2\cdot 1-1}=1$$

Because $\lim_{x\to 1^-} f(x)=-1$ and $\lim_{x\to 1^+} f(x)=1$ are not equal, $\lim_{x\to 1} f(x)$ does not exist.

✎ Pencil Problem #3 ✐

3. Consider the piecewise function

$$f(x)=\begin{cases}x^2+5 & \text{if } x<2\\ x^3+1 & \text{if } x\geq 2.\end{cases}$$

Find $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$, or state that the limit does not exist.

Objective #4: Find limits of fractional expressions in which the limit of the denominator is zero.

✔ Solved Problem #4

4a. Find $\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$.

First find the limit of the denominator as $x \to 1$.

$$\lim_{x \to 1}(x-1) = 1 - 1 = 0$$

Since the limit of the denominator is 0, we cannot use the rule for the limit of a quotient. We factor the numerator and cancel a common factor and then proceed to apply limit properties.

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x+3)(x-1)}{x-1}$$
$$= \lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{x-1}}$$
$$= \lim_{x \to 1}(x+3)$$
$$= 1 + 3 = 4$$

4b. Find $\lim_{x \to 0} \frac{\sqrt{9+x} - 3}{x}$.

As x approaches 0, the denominator of the expression approaches 0. Thus, the quotient property for limits cannot be used. We multiply the numerator and denominator by $\sqrt{9+x} + 3$ to eliminate the radical in the numerator and then simplify.

$$\lim_{x \to 0} \frac{\sqrt{9+x} - 3}{x} = \lim_{x \to 0} \left[\frac{\sqrt{9+x} - 3}{x} \cdot \frac{\sqrt{9+x} + 3}{\sqrt{9+x} + 3} \right]$$
$$= \lim_{x \to 0} \frac{(\sqrt{9+x})^2 - 3^2}{x(\sqrt{9+x} + 3)}$$
$$= \lim_{x \to 0} \frac{9 + x - 9}{x(\sqrt{9+x} + 3)}$$
$$= \lim_{x \to 0} \frac{x}{x(\sqrt{9+x} + 3)}$$
$$= \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{9+x} + 3)}$$
$$= \lim_{x \to 0} \frac{1}{\sqrt{9+x} + 3}$$
$$= \frac{1}{\sqrt{9+0} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

✎ Pencil Problem #4 ✎

4a. Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

4b. Find $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 8 *(11.2 #1)* **1b.** –6 *(11.2 #2)* **1c.** 2 *(11.2 #3)*

2a. 14 *(11.2 #5)* **2b.** 28 *(11.2 #7)* **2c.** 6 *(11.2 #9)* **2d.** 8 *(11.2 #11)*

2e. 5 *(11.2 #15)* **2f.** 3 *(11.2 #19)*

3. $\lim_{x \to 2^-} f(x) = 9$; $\lim_{x \to 2^+} f(x) = 9$; $\lim_{x \to 2} f(x) = 9$ *(11.2 #45)*

4a. 2 *(11.2 #21)* **4b.** $\frac{1}{2}$ *(11.2 #29)*

Section 11.3
Limits and Continuity

Taxing Your Limits

Did you know that as your income increases so does the amount that you pay in income taxes? But not only does the amount you pay in taxes increase but also the percentage of your income that you pay in taxes increases. So as you earn greater amounts of money, your tax liability increases at a greater rate.

In the Exercise Set, you will work with a piecewise function that describes the amount of taxes owed by a single taxpayer. You will explore whether or not there is a "big jump" in the amount of taxes owed when you move up to the next tax bracket.

Objective #1: Determine whether a function is continuous at a number.

✔ *Solved Problem #1*

1a. Determine whether the function $f(x)=\dfrac{x-2}{x^2-4}$ is continuous at 1.

Is $f(1)$ defined? Yes.

$$f(1)=\frac{1-2}{1^2-4}=\frac{-1}{-3}=\frac{1}{3}$$

Does $\lim\limits_{x\to 1} f(x)$ exist? Yes.

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x-2}{x^2-4}=\frac{\lim\limits_{x\to 1}(x-2)}{\lim\limits_{x\to 1}(x^2-4)}=\frac{1-2}{1^2-4}=\frac{-1}{-3}=\frac{1}{3}$$

Does $\lim\limits_{x\to 1} f(x)=f(1)$? Yes.

$f(1)$ and $\lim\limits_{x\to 1} f(x)$ both equal $\dfrac{1}{3}$.

Because the three conditions are satisfied, we conclude that f is continuous at 1.

1b. Determine whether the function $f(x)=\dfrac{x-2}{x^2-4}$ is continuous at 2.

Substituting 2 into the denominator of the expression results in 0. Since division by 0 is undefined, 2 is not in the domain of the function. Since f is not defined at 2, f is not continuous at 2.

✎ *Pencil Problem #1* ✎

1a. Determine whether the function $f(x)=\dfrac{x-5}{x+5}$ is continuous at 5.

1b. Determine whether the function $f(x)=\dfrac{x+5}{x-5}$ is continuous at 5.

Objective #2: Determine for what numbers a function is discontinuous.

✔ *Solved Problem #2*

2. Determine for what numbers, x, if any, the function f is discontinuous.

$$f(x)=\begin{cases}2x & \text{if } x\le 0\\ x^2+1 & \text{if } 0<x\le 2\\ 7-x & \text{if } x>2\end{cases}$$

Since each piece of the function is defined by a polynomial, the only possible discontinuities are at $x = 0$ and $x = 2$. We check for continuity at each of these x-values.

At $x = 0$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x = 2\cdot 0 = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^2+1) = 0^2+1 = 1$$

Since $\lim_{x\to 0^-} f(x) \ne \lim_{x\to 0^+} f(x)$, $\lim_{x\to 0} f(x)$ does not exist.

Since the second condition for continuity is not satisfied, f is discontinuous at 0.

At $x = 2$:

$f(2) = 2^2+1 = 5$, so f is defined at 2.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2+1) = 2^2+1 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (7-x) = 7-2 = 5$$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 5$, $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2} f(x) = 5$.

Finally, $\lim_{x\to 2} f(x) = 5 = f(2)$.

Since all three conditions for continuity are satisfied, f is continuous at 2.

The only value of x for which f is discontinuous is 0.

✎ *Pencil Problem #2* ✐

2. Determine for what numbers, x, if any, the function f is discontinuous.

$$f(x)=\begin{cases}x+6 & \text{if } x\le 0\\ 6 & \text{if } 0<x\le 2\\ x^2+1 & \text{if } x>2\end{cases}$$

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. continuous *(11.3 #9)* **1b.** not continuous *(11.3 #7)*

2. discontinuous at 2 *(11.3 #31)*

Section 11.4
Introduction to Derivatives

Measuring Change

We all know that things change. In this section, we combine two topics we've already studied, average rate of change and limits, to study how quickly things change. By applying limits to expressions representing average rate of change, we can define *instantaneous rate of change*. If an object is thrown upward and we know its initial velocity and the height from which it is thrown, we can write a function that describes the height of the object until it hits the ground. When we apply the ideas in this section to that function, we can also determine the velocity of the object at each instant until it hits the ground.

Objective #1: Find slopes and equations of tangent lines.

✔ *Solved Problem #1*

1a. Find the slope of the tangent line to the graph of $f(x) = x^2 - x$ at (4, 12).

We use the formula $m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, with $a = 4$, the x-coordinate of the given point on the graph of f. After simplifying the expressions in the numerator, we factor h from the numerator and divide both the numerator and denominator by h to eliminate the factor of h in the denominator.

$$\begin{aligned}
m_{\tan} &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} \\
&= \lim_{h \to 0} \frac{[(4+h)^2 - (4+h)] - [4^2 - 4]}{h} \\
&= \lim_{h \to 0} \frac{[16 + 8h + h^2 - 4 - h] - [16 - 4]}{h} \\
&= \lim_{h \to 0} \frac{[12 + 7h + h^2] - 12}{h} \\
&= \lim_{h \to 0} \frac{7h + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(7+h)}{h} \\
&= \lim_{h \to 0} (7+h) = 7 + 0 = 7
\end{aligned}$$

Thus, the slope of the tangent line to the graph of $f(x) = x^2 - x$ at (4, 12) is 7.

✎ *Pencil Problem #1* ✐

1a. Find the slope of the tangent line to the graph of $f(x) = x^2 + 4$ at (−1, 5).

1b. Find the slope-intercept equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at (1, 1).

We begin by finding the slope of the tangent line. Note that we use the method of rationalizing the numerator in order to evaluate the limit. Along the way, we factor *h* from the numerator and divide both the numerator and denominator by *h* to eliminate the factor of *h* in the denominator.

$$\begin{aligned}
m_{\tan} &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{1+h}-\sqrt{1}}{h} \\
&= \lim_{h\to 0}\left[\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right] \\
&= \lim_{h\to 0}\frac{(\sqrt{1+h})^2-1^2}{h(\sqrt{1+h}+1)} \\
&= \lim_{h\to 0}\frac{1+h-1}{h(\sqrt{1+h}+1)} \\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{1+h}+1)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{1+1}=\frac{1}{2}
\end{aligned}$$

We now use the slope just found and the given point, (1, 1), to write the equation of the tangent line in point-slope form. Then we solve for *y* to write the equation in slope intercept form.

$$\begin{aligned}
y-y_1 &= m(x-x_1) \\
y-1 &= \frac{1}{2}(x-1) \\
y-1 &= \frac{1}{2}x-\frac{1}{2} \\
y &= \frac{1}{2}x+\frac{1}{2}
\end{aligned}$$

1b. Find the slope-intercept equation of the tangent line to the graph of $f(x)=\sqrt{x}$ at (9, 3).

Objective #2: Find the derivative of a function.

✔ *Solved Problem #2*

2a. Find the derivative of $f(x) = x^2 - 5x$ at x. That is, find $f'(x)$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{[(x+h)^2 - 5(x+h)] - [x^2 - 5x]}{h}$$
$$= \lim_{h\to 0}\frac{[x^2 + 2xh + h^2 - 5x - 5h] - x^2 + 5x}{h}$$
$$= \lim_{h\to 0}\frac{2xh + h^2 - 5h}{h}$$
$$= \lim_{h\to 0}\frac{h(2x+h-5)}{h}$$
$$= \lim_{h\to 0}(2x+h-5) = 2x+0-5 = 2x-5$$

The derivative is $f'(x) = 2x - 5$.

2b. Find the slope of the tangent line to the graph of $f(x) = x^2 - 5x$ at $x = -1$.

We found $f'(x)$ in Solved Problem #2a. The slope of the tangent line is $f'(-1)$.

$f'(-1) = 2(-1) - 5 = -2 - 5 = -7$

The slope of the tangent line at $x = -1$ is -7.

✎ *Pencil Problem #2* ✎

2a. Find the derivative of $f(x) = x^2 - 3x + 5$ at x. That is, find $f'(x)$.

2b. Find the slope of the tangent line to the graph of $f(x) = x^2 - 3x + 5$ at $x = 2$.

Objective #3: Find average and instantaneous rates of change.

✔ *Solved Problem #3*

3a. The function $f(x) = x^3$ describes the volume of a cube, $f(x)$, in cubic inches, whose length, width, and height each measure x inches. If x is changing, find the average rate of change of the volume with respect to x as x changes from 4 inches to 4.1 inches and from 4 inches to 4.01 inches.

We use the difference quotient for both calculations.
From 4 inches to 4.1 inches:

$$\frac{f(4.1)-f(4)}{4.1-4} = \frac{4.1^3 - 4^3}{0.1}$$
$$= \frac{68.921-64}{0.1}$$
$$= \frac{4.921}{0.1}$$
$$= 49.21$$

(continued on next page)

✎ *Pencil Problem #3* ✎

3a. The function $f(x) = x^2$ describes the area of a square, $f(x)$, in square inches, whose sides each measure x inches. If x is changing, find the average rate of change of the area with respect to x as x changes from 6 inches to 6.1 inches and from 6 inches to 6.01 inches.

The average rate of change is 49.21 cubic inches per inch as x changes from 4 inches to 4.1 inches.

From 4 inches to 4.01 inches:

$$\begin{aligned}\frac{f(4.01)-f(4)}{4.01-4} &= \frac{4.01^3-4^3}{0.01}\\ &= \frac{64.481201-64}{0.01}\\ &= \frac{0.481201}{0.01}\\ &= 48.1201\end{aligned}$$

The average rate of change is 48.1201 cubic inches per inch as x changes from 4 inches to 4.01 inches.

3b. The function $f(x) = x^3$ describes the volume of a cube, $f(x)$, in cubic inches, whose length, width, and height each measure x inches. If x is changing, find the instantaneous rate of change of the volume with respect to x when $x = 4$ inches.

The instantaneous rate of change is the derivative. We begin by finding $f'(x)$ and then find $f'(4)$.

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{[x^3+3x^2h+3xh^2+h^3]-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}\\ &= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2+3x\cdot 0+0^2 = 3x^2\end{aligned}$$

Since $f'(x) = 3x^2$, we have $f'(4) = 3(4)^2 = 48$.

The instantaneous rate of change of the volume with respect to x is 48 cubic inches per inch when $x = 4$ inches.

3b. The function $f(x) = x^2$ describes the area of a square, $f(x)$, in square inches, whose sides each measure x inches. If x is changing, find the instantaneous rate of change of the area with respect to x when $x = 6$ inches.

Objective #4: Find instantaneous velocity.

✔ *Solved Problem #4*

4a. A ball is thrown straight up from ground level with an initial velocity of 96 feet per second. The function $s(t) = -16t^2 + 96t$ describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is thrown. What is the instantaneous velocity of the ball after 4 seconds?

The instantaneous velocity is the derivative.

$$
\begin{aligned}
& s'(a) \\
&= \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} \\
&= \lim_{h \to 0} \frac{[-16(a+h)^2 + 96(a+h)] - [-16a^2 + 96a]}{h} \\
&= \lim_{h \to 0} \frac{[-16(a^2 + 2ah + h^2) + 96(a+h)] - [-16a^2 + 96a]}{h} \\
&= \lim_{h \to 0} \frac{-16a^2 - 32ah - 16h^2 + 96a + 96h + 16a^2 - 96a}{h} \\
&= \lim_{h \to 0} \frac{-32ah - 16h^2 + 96h}{h} \\
&= \lim_{h \to 0} \frac{h(-32a - 16h + 96)}{h} \\
&= \lim_{h \to 0} (-32a - 16h + 96) \\
&= -32a - 16 \cdot 0 + 96 \\
&= -32a + 96
\end{aligned}
$$

We now compute $s'(4)$.

$s'(4) = -32(4) + 96 = -128 + 96 = -32$

The instantaneous velocity of the ball after 4 seconds is −32 feet per second. The negative velocity indicates that the ball is moving downward at this point.

✎ *Pencil Problem #4* ✐

4a. A ball is thrown straight up from ground level with an initial velocity of 64 feet per second. The function $s(t) = -16t^2 + 64t$ describes the ball's height above the ground, $s(t)$, in feet, t seconds after it is thrown. What is the instantaneous velocity of the ball after 3 seconds?

4b. Use the information in Solved Problem #4a to determine the instantaneous velocity of the ball when it hits the ground.

We need to know the time when the ball hits the ground. Solve $s(t) = 0$.

$$-16t^2 + 96t = 0$$
$$-16t(t-6) = 0$$
$$-16t = 0 \quad \text{or} \quad t - 6 = 0$$
$$t = 0 \qquad\qquad t = 6$$

Since the ball is thrown at $t = 0$, we are not interested in this value of t. The ball hits the ground after 6 seconds. Use the derivative found in Solved Problem #4a to find $s'(6)$.

$$s'(6) = -32(6) + 96 = -192 + 96 = -96$$

The instantaneous velocity of the ball when it hits the ground is –96 feet per second.

4b. Use the information in Pencil Problem #4a to determine the instantaneous velocity of the ball when it hits the ground.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. -2 *(11.4 #3)* **1b.** $y = \frac{1}{6}x + \frac{3}{2}$ *(11.4 #11)*

2a. $f'(x) = 2x - 3$ **2b.** 1 *(11.4 #19)*

3a. 12.1 square inches per inch; 12.01 square inches per inch *(11.4 #37a)*

3b. 12 square inches per inch *(11.4 #37b)*

4a. –32 feet per second **4b.** –64 feet per second *(11.4 #43)*