Section 10.1
Sequences and Summation Notation

Bees, Trees, and Piano Keys !

What can those three things possibly have in common?

In this section, we will study sequences. One amazing example is called the Fibonacci sequence, an infinite sequence of numbers investigated by Leonardo of Pisa, also known as Fibonacci, an Italian mathematician of the thirteenth century.

The sequence is generated using simple addition, and yet it shows up in some unexpected, and complex, ways.

As you read the textbook, you will find interesting areas where these concepts apply.

Objective #1: Find particular terms of a sequence from the general term.

Solved Problem #1

1a. Write the first four terms of the sequence whose *n*th term, or general term, is $a_n = 2n + 5$.

$$a_n = 2n + 5$$
$$a_1 = 2(1) + 5 = 7$$
$$a_2 = 2(2) + 5 = 9$$
$$a_3 = 2(3) + 5 = 11$$
$$a_4 = 2(4) + 5 = 13$$

The first four terms are 7, 9, 11, and 13.

1b. Write the first four terms of the sequence whose *n*th term, or general term, is $a_n = \frac{(-1)^n}{2^n + 1}$.

$$a_n = \frac{(-1)^n}{2^n + 1}$$
$$a_1 = \frac{(-1)^1}{2^1 + 1} == \frac{-1}{3} - \frac{1}{3}$$
$$a_2 = \frac{(-1)^2}{2^2 + 1} = \frac{1}{5}$$
$$a_3 = \frac{(-1)^3}{2^3 + 1} = \frac{-1}{9} = -\frac{1}{9}$$
$$a_4 = \frac{(-1)^4}{2^4 + 1} = \frac{1}{17}$$

The first four terms are $-\frac{1}{3}, \frac{1}{5}, -\frac{1}{9},$ and $\frac{1}{17}$.

Pencil Problem #1

1a. Write the first four terms of the sequence whose *n*th term, or general term, is $a_n = 3n + 2$.

1b. Write the first four terms of the sequence whose *n*th term, or general term, is $a_n = (-1)^n (n + 3)$.

Objective #2: Use recursion formulas.

✔ *Solved Problem #2*

2. Find the first four terms of the sequence in which $a_1 = 3$ and $a_n = 2a_{n-1} + 5$ for $n \geq 2$.

$a_1 = 3$
$a_2 = 2a_1 + 5 = 2(3) + 5 = 11$
$a_3 = 2a_2 + 5 = 2(11) + 5 = 27$
$a_4 = 2a_3 + 5 = 2(27) + 5 = 59$

The first four terms are 3, 11, 27, and 59.

✎ *Pencil Problem #2* ✎

2. Find the first four terms of the sequence in which $a_1 = 4$ and $a_n = 2a_{n-1} + 3$ for $n \geq 2$.

Objective #3: Use factorial notation.

✔ *Solved Problem #3*

3. Write the first four terms of the sequence whose *n*th term is $a_n = \dfrac{20}{(n+1)!}$.

$$a_n = \frac{20}{(n+1)!}$$
$$a_1 = \frac{20}{(1+1)!} = \frac{20}{2!} = 10$$
$$a_2 = \frac{20}{(2+1)!} = \frac{20}{3!} = \frac{20}{6} = \frac{10}{3}$$
$$a_3 = \frac{20}{(3+1)!} = \frac{20}{4!} = \frac{20}{24} = \frac{5}{6}$$
$$a_4 = \frac{20}{(4+1)!} = \frac{20}{5!} = \frac{20}{120} = \frac{1}{6}$$

The first four terms are $10, \frac{10}{3}, \frac{5}{6}$, and $\frac{1}{6}$.

✎ *Pencil Problem #3* ✎

3. Write the first four terms of the sequence whose *n*th term is $a_n = \dfrac{n^2}{n!}$.

Objective #4: Use summation notation.

✔ *Solved Problem #4*

4a. Expand and evaluate the sum: $\displaystyle\sum_{k=3}^{5}\left(2^k - 3\right)$.

$$\sum_{k=3}^{5}\left(2^k - 3\right)$$
$$= \left(2^3 - 3\right) + \left(2^4 - 3\right) + \left(2^5 - 3\right)$$
$$= (8 - 3) + (16 - 3) + (32 - 3)$$
$$= 5 + 13 + 29$$
$$= 47$$

✎ *Pencil Problem #4* ✎

4a. Expand and evaluate the sum: $\displaystyle\sum_{k=1}^{5} k(k+4)$.

4b. Expand and evaluate the sum: $\sum_{i=1}^{5} 4.$

$$\sum_{i=1}^{5} 4 = 4+4+4+4+4$$
$$= 20$$

4b. Expand and evaluate the sum: $\sum_{i=5}^{9} 11$

4c. Express the sum using summation notation. Use 1 as the lower limit of summation and *i* for the index of summation.

$$1^2+2^2+3^2+\cdots+9^2$$

The sum has nine terms, each of the form i^2, starting at $i=1$ and ending at $i=9$.

$$1^2+2^2+3^2+\cdots+9^2=\sum_{i=1}^{9} i^2$$

4c. Express the sum using summation notation. Use 1 as the lower limit of summation and *i* for the index of summation.

$$2+2^2+2^3+\ldots+2^{11}$$

4d. Express the sum using summation notation. Use 1 as the lower limit of summation and *i* for the index of summation.

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}$$

The sum has *n* terms, each of the form $\frac{1}{2^{i-1}}$, starting at $i=1$ and ending at $i=n$.

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}=\sum_{i=1}^{n}\frac{1}{2^{i-1}}$$

4d. Express the sum using summation notation. Use 1 as the lower limit of summation and *i* for the index of summation.

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{14}{14+1}$$

<u>Answers</u> <u>for Pencil Problems *(Textbook Exercise references in parentheses)*</u>:

1a. 5, 8, 11, 14 *(10.1 #1)*

1b. –4, 5, –6, 7 *(10.1 #7)*

2. 4, 11, 25, 53 *(10.1 #17)*

3. $1, 2, \frac{3}{2}, \frac{2}{3}$ *(10.1 #19)*

4a. 115 *(10.1 #33)*

4b. 55 *(10.1 #37)*

4c. $\sum_{i=1}^{11} 2^i$ *(10.1 #45)*

4d. $\sum_{i=1}^{14} \frac{i}{i+1}$ *(10.1 #49)*

Section 10.2
Arithmetic Sequences

IT'S A FULL THEATER TONIGHT!

Some theaters have the same number of seats in each row. But other theaters are more fan-shaped.

In this section of the textbook, we will encounter such a fan-shaped theater, and we will use the techniques of this section to quickly determine the total number of seats without actually adding the number in each row.

Objective #1: Find the common difference for an arithmetic sequence.

✔ ***Solved Problem #1***

1. True or false: An arithmetic sequence is a sequence in which each term after the first differs from the preceding term by a constant amount.

true

✎ ***Pencil Problem #1*** ✐

1. True or false: In an arithmetic sequence, each term after the first term can be obtained by adding the common difference to the preceding term.

Objective #2: Write terms of an arithmetic sequence.

✔ ***Solved Problem #2***

2. Write the first six terms of the arithmetic sequence with first term 100 and common difference -30.

$a_1 = 100$

$a_2 = 100 + (-30) = 70$

$a_3 = 70 + (-30) = 40$

$a_4 = 40 + (-30) = 10$

$a_5 = 10 + (-30) = -20$

$a_6 = -20 + (-30) = -50$

✎ ***Pencil Problem #2*** ✐

2. Write the first six terms of the arithmetic sequence with first term -7 and common difference 4.

Objective #3: Use the formula for the general term of an arithmetic sequence.

✔ Solved Problem #3

3a. Find the ninth term of the arithmetic sequence whose first term is 6 and whose common difference is -5.

$a_1 = 6,\ d = -5$

To find the ninth term, a_9, replace n in the formula with 9, replace a_1 with 6, and replace d with –5.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_9 &= 6 + (9-1)(-5) \\ &= 6 + 8(-5) \\ &= 6 + (-40) \\ &= -34 \end{aligned}$$

3b. In 2010, 16% of the U.S. population was Latino. On average, this is projected to increase by approximately 0.35% per year. Write a formula for the *n*th term of the arithmetic sequence that describes the percentage of the U.S. population that will be Latino n years after 2009.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= 16 + (n-1)0.35 \\ &= 0.35n + 15.65 \end{aligned}$$

3c. Use the result from the previous problem to project the percentage of the U.S. population that will be Latino in 2030.

2030 is 21 years after 2009.

$$\begin{aligned} a_n &= 0.35n + 15.65 \\ a_{20} &= 0.35(21) + 15.65 = 23 \end{aligned}$$

In 2030, 23% of the U.S. population is projected to be Latino.

✎ Pencil Problem #3 ✎

3a. Find the 50th term of the arithmetic sequence whose first term is 7 and whose common difference is 5.

3b. In 1970, 11.0% of Americans ages 25 and older had completed four years of college or more. On average, this percentage has increased by approximately 0.5 each year. Write a formula for the *n*th term of the arithmetic sequence that models the percentage of Americans ages 25 and older who had or will have completed four years of college or more n years after 1969.

3c. Use the result from the previous problem to project the percentage of Americans ages 25 and older who will have completed four years of college or more by 2019.

Objective #4: Use the formula for the sum of the first n terms of an arithmetic sequence.

✔ Solved Problem #4

4a. Find the sum of the first 15 terms of the arithmetic sequence: 3, 6, 9, 12, ...

To find the sum of the first 15 terms, S_{15}, replace n in the formula with 15.

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{15} = \frac{15}{2}(a_1 + a_{15})$$

Use the formula for the general term of a sequence to find a_{15}. The common difference, d, is 3, and the first term, a_1, is 3.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{15} &= 3 + (15-1)(3) \\ &= 3 + 14(3) \\ &= 3 + 42 \\ &= 45 \end{aligned}$$

Thus, $S_{15} = \frac{15}{2}(3+45) = \frac{15}{2}(48) = 360$.

4b. Find the following sum: $\sum_{i=1}^{30}(6i-11)$.

$$\begin{aligned} &\sum_{i=1}^{30}(6i-11) \\ &= (6\cdot 1 - 11) + (6\cdot 2 - 11) + (6\cdot 3 - 11) + \ldots + (6\cdot 30 - 11) \\ &= -5 + 1 + 7 + \ldots + 169 \end{aligned}$$

The first term, a_1, is –5.
The common difference, d, is $1-(-5)=6$.
The last term, a_{30}, is 169.

$$\begin{aligned} S_n &= \tfrac{n}{2}(a_1 + a_n) \\ S_{30} &= \tfrac{30}{2}(-5+169) \\ &= 15(164) \\ &= 2460 \end{aligned}$$

Thus, $\sum_{i=1}^{30}(6i-11) = 2460$

✎ Pencil Problem #4 ✐

4a. Find the sum of the first 50 terms of the arithmetic sequence: $-10, -6, -2, 2, \ldots$

4b. Find the following sum: $\sum_{i=1}^{100} 4i$.

4c. The model $a_n = 1800n + 64{,}130$ describes yearly adult residential community costs n years after 2013. How much would it cost for the adult residential community for a ten-year period beginning in 2014?

$$a_n = 1800n + 64{,}130$$
$$a_1 = 1800(1) + 64{,}130 = 65{,}930$$
$$a_{10} = 1800(10) + 64{,}130 = 82{,}130$$

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$
$$\begin{aligned} S_{10} &= \frac{10}{2}\left(a_1 + a_{10}\right) \\ &= 5\left(65{,}930 + 82{,}130\right) \\ &= 5\left(148{,}060\right) \\ &= \$740{,}300 \end{aligned}$$

It would cost $740,300 for the ten-year period beginning in 2014.

4c. A section in a stadium has 20 seats in the first row, 23 seats in the second row, increasing by 3 seats each row for a total of 38 rows. How many seats are in this section of the stadium?

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. true *(10.2 #1)* **2.** $-7, -3, 1, 5, 9, 13$ *(10.2 #3)*

3a. 252 *(10.2 #17)* **3b.** $a_n = 0.5n + 10.5$ *(10.2 #61a)* **3c.** 35.5% *(10.2 #61b)*

4a. 4400 *(10.2 #37)* **4b.** 20,200 *(10.2 #49)* **4c.** 2869 seats *(10.2 #71)*

Section 10.3
Geometric Sequences and Series

How Much Will You End Up With?

Suppose you are 24 and you have just landed a job!

You decide that you can save for retirement by putting aside $80 per month into an account which pays 5% compounded monthly.

What will the account balance be when you reach age 65?

In this section, several applications will deal with money and we will apply geometric sequences and series to find answers to a variety of financial questions.

Objective #1: Find the common ratio of a geometric sequence.

✔ *Solved Problem #1*

1. True or False: The sequence

$$6, -12, 24, -48, 96, ...$$

is an example of a geometric sequence.

True. Each term after the first term is -2 times the previous term. The common ration is -2.

✎ *Pencil Problem #1* ✎

1. True or False: The sequence

$$2, 6, 24, 120, \ldots$$

is an example of a geometric sequence.

Objective #2: Write terms of a geometric sequence.

✔ *Solved Problem #2*

2. Write the first six terms of the geometric sequence with first term 12 and common ratio $\frac{1}{2}$.

$$a_1 = 12,\ r = \frac{1}{2}$$

$$a_2 = 12\left(\tfrac{1}{2}\right)^1 = 6$$

$$a_3 = 12\left(\tfrac{1}{2}\right)^2 = \tfrac{12}{4} = 3$$

$$a_4 = 12\left(\tfrac{1}{2}\right)^3 = \tfrac{12}{8} = \tfrac{3}{2}$$

$$a_5 = 12\left(\tfrac{1}{2}\right)^4 = \tfrac{12}{16} = \tfrac{3}{4}$$

$$a_6 = 12\left(\tfrac{1}{2}\right)^5 = \tfrac{12}{32} = \tfrac{3}{8}$$

The first six terms are $12, 6, 3, \frac{3}{2}, \frac{3}{4}$, and $\frac{3}{8}$.

✎ *Pencil Problem #2* ✎

2. Write the first five terms of the geometric sequence with first term 5 and common ratio 3.

Objective #3: Use the formula for the general term of a geometric sequence.

✔ ***Solved Problem #3***

3a. Find the seventh term of the geometric sequence whose first term is 5 and whose common ratio is -3.

$a_1 = 5,\ r = -3$

$a_n = a_1 r^{n-1}$

$a_7 = 5(-3)^{7-1}$

$= 5(-3)^6$

$= 5(729)$

$= 3645$

The seventh term is 3645.

3b. Write the general term for the geometric sequence: $3, 6, 12, 24, 48, \ldots$
Then use the formula for the general term to find the eighth term.

$r = \frac{6}{3} = 2,\ a_1 = 3$

Formula for the general term:

$a_n = a_1(r)^{n-1}$

$a_n = 3(2)^{n-1}$

Find the eighth term:

$a_n = 3(2)^{n-1}$

$a_8 = 3(2)^{8-1}$

$= 3(2)^7$

$= 3(128)$

$= 384$

The eighth term is 384.

✎ ***Pencil Problem #3*** ✐

3a. Find the eighth term of the geometric sequence whose first term is 6 and whose common ratio is 2.

3b. Write the general term for the geometric sequence: $3, 12, 48, 192, \ldots$
Then use the formula for the general term to find the seventh term.

Objective #4: Use the formula for the sum of the first *n* terms of a geometric sequence.

✔ Solved Problem #4

4a. Find the sum of the first nine terms of the geometric sequence: $2, -6, 18, -54, \ldots$

$$a_1 = 2,\ r = \frac{-6}{2} = -3$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_9 = \frac{2\left(1-(-3)^9\right)}{1-(-3)} = \frac{2(19{,}684)}{4} = 9842$$

The sum of the first nine terms is 9842.

4b. Find the following sum: $\sum_{i=1}^{8} 2\cdot 3^i$.

$$a_1 = 2\cdot(3)^1 = 6,\ r = 3$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_8 = \frac{6\left(1-3^8\right)}{1-3} = \frac{6(-6560)}{-2} = 19{,}680$$

Thus, $\sum_{i=1}^{8} 2\cdot 3^i = 19{,}680$.

✎ Pencil Problem #4 ✎

4a. Find the sum of the first 11 terms of the geometric sequence: $3, -6, 12, -24, \ldots$

4b. Find the following sum: $\sum_{i=1}^{10} 5\cdot 2^i$.

4c. A job pays a salary of \$30,000 the first year. During the next 29 years, the salary increases by 6% each year. What is the total lifetime salary over the 30-year period? Round to the nearest dollar.

$$a_1 = 30{,}000,\ r = 1.06$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S_{30} = \frac{30{,}000\left(1-(1.06)^{30}\right)}{1-1.06} \approx 2{,}371{,}746$$

The total lifetime salary is \$2,371,746.

4c. A job pays a salary of \$24,000 the first year. During the next 19 years, the salary increases by 5% each year. What is the total lifetime salary over the 20-year period? Round to the nearest dollar.

Objective #5: Find the value of an annuity.

✔ *Solved Problem #5*

5. At age 30, to save for retirement, you decide to deposit \$100 at the end of each month into an IRA that pays 9.5% compounded monthly. Find how much will you have in the IRA when you retire at age 65 and find how much is interest.

$$A = \frac{P\left[\left(1+\frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$$

$$P = 100,\ r = 0.095,\ n = 12,\ t = 35$$

$$A = \frac{100\left[\left(1+\frac{0.095}{12}\right)^{12\cdot 35} - 1\right]}{\frac{0.095}{12}} \approx 333{,}946$$

The value of the IRA will be \$333,946.

Find the interest:

$$\begin{aligned}\text{Interest} &= \text{Value of IRA} - \text{Total deposits}\\ &\approx \$333{,}946 - \$100\cdot 12\cdot 35\\ &\approx \$333{,}946 - \$42{,}000\\ &\approx \$291{,}946\end{aligned}$$

✎ *Pencil Problem #5* ✎

5. At age 25, to save for retirement, you decide to deposit \$50 at the end of each month into an IRA that pays 5.5% compounded monthly. Find how much will you have in the IRA when you retire at age 65 and find how much is interest.

Objective #6: Use the formula for the sum of an infinite geometric series.

✔ Solved Problem #6

6a. Find the sum of the infinite geometric series:

$$3+2+\frac{4}{3}+\frac{8}{9}+\cdots$$

$$a_1=3,\ r=\frac{2}{3}$$

$$S=\frac{a_1}{1-r}$$

$$S=\frac{3}{1-\frac{2}{3}}$$

$$=\frac{3}{\frac{1}{3}}$$

$$=9$$

The sum of this infinite geometric series is 9.

6b. Express $0.\overline{9}$ as a fraction in lowest terms.

$$0.\overline{9}=0.9999\cdots=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$$

$$a_1=\frac{9}{10}, r=\frac{1}{10}$$

$$S=\frac{a_1}{1-r}$$

$$S=\frac{\frac{9}{10}}{1-\frac{1}{10}}$$

$$=\frac{\frac{9}{10}}{\frac{9}{10}}$$

$$=1$$

An equivalent fraction for $0.\overline{9}$ is 1.

✎ Pencil Problem #6 ✐

6a. Find the sum of the infinite geometric series:

$$3+\frac{3}{4}+\frac{3}{4^2}+\frac{3}{4^3}+\cdots$$

6b. Express $0.\overline{5}$ as a fraction in lowest terms.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. false *(10.3 #105)*

2. $5, 15, 45, 135, 405$ *(10.3 #1)*

3a. 768 *(10.3 #9)* **3b.** $a_n = 3(4)^{n-1}$; $a_7 = 12{,}288$ *(10.3 #17)*

4a. 2049 *(10.3 #29)* **4b.** 10,230 *(10.3 #33)* **4c.** $793,583 *(10.3 #73)*

5. $87,052; $63,052 *(10.3 #79)*

6a. 4 *(10.3 #39)* **6b.** $\frac{5}{9}$ *(10.3 #45)*

Section 10.4
Mathematical Induction

Will They ALL Fall Down?

The mathematical principle of this section can be illustrated using an unending line of dominoes. If the first domino is pushed over, it knocks down the next, which knocks down the next, and so on, in a chain reaction.

To topple all the dominoes in the infinite sequence, two conditions must be satisfied:

1. The first domino must be knocked down.

2. If the domino in position k is knocked down, then the domino in position $k+1$ must be knocked down.

If the second condition is not satisfied, it does not follow that all the dominoes will topple. For example, suppose the dominoes are spaced far enough apart so that a falling domino does not push over the next domino in the line.

Objective #1: Understand the principle of mathematical induction.

✔ *Solved Problem #1*

1a. For the given statement S_n, write the statement S_1.

$$S_n: 2+4+6+\cdots+2n = n(n+1)$$

If $n=1$ then the statement S_1 is obtained by writing the first term, 2, on the left, and substituting 1 for n on the right.

$S_1: 2 = 1(1+1)$

✎ *Pencil Problem #1* ✎

1a. For the given statement S_n, write the statement S_1.

$$S_n: 1+3+5+(2n-1) = n^2$$

✔ *Solved Problem #1b*

1b. For the given statement S_n, write the two statements S_k, and S_{k+1}.

$$S_n : 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4}$$

Write S_k by taking the sum of the first *k* terms on the left and replacing *n* with *k* on the right.

$$S_k : 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2 (k+1)^2}{4}$$

Write S_{k+1} by taking the sum of the first $k+1$ terms on the left and replacing *n* with $k+1$ on the right.

$$S_{k+1} : 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2 (k+1+1)^2}{4}$$
$$= \frac{(k+1)^2 (k+2)^2}{4}$$

✎ *Pencil Problem #1b* ✐

1b. For the given statement S_n, write the two statements S_k, and S_{k+1}.

$$S_n : 3 + 7 + 11 + \cdots + (4n-1) = n(2n+1)$$

Objective #2: Prove statements using mathematical induction.

✔ ***Solved Problem #2***

2. Use mathematical induction to prove that $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ for all positive integers *n*.

Step 1. *Show that* S_1 *is true*:

$$1^3=\frac{1^2(1+1)^2}{4}$$
$$1=\frac{1(2)^2}{4}$$
$$1=\frac{4}{4}$$
$$1=1,\ \text{True}$$

Step 2. *Show that if* S_k *is true, then* S_{k+1} *is true*:

Assume $1^3+2^3+3^3+\cdots+k^3=\dfrac{k^2(k+1)^2}{4}$ is true. Then,

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$$
$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}$$
$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$
$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2\left(k^2+4(k+1)\right)}{4}$$
$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2\left(k^2+4k+4\right)}{4}$$
$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{(k+1)^2(k+2)^2}{4}$$

The final statement is S_{k+1}.

Thus, by mathematical induction, the statement $1^3+2^3+3^3+\cdots+n^3=\dfrac{n^2(n+1)^2}{4}$ is true for all positive integers *n*.

✎ *Pencil Problem #2* ✐

2. Use mathematical induction to prove that $1+2+2^2+\cdots+2^{n-1}=2^n-1$ for all positive integers *n*.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. $S_1: 1=1^2$ *(10.4 #1)*

1b. $S_k: 3+7+11+\cdots+(4k-1)=k(2k+1)$

$S_{k+1}: 3+7+11+\cdots+\left[4(k+1)-1\right]=(k+1)\left[2(k+1)+1\right]=(k+1)(2k+3)$ *(10.4 #7)*

2. *Show that* S_1 *is true*:
$$1=2^1-1$$
$$1=2-1$$
$$1=1,\ \text{true}$$

Show that if S_k *is true, then* S_{k+1} *is true*:

Assume $S_k: 1+2+2^2+\cdots+2^{k-1}=2^k-1$ is true. Then, $1+2+2^2+\cdots+2^{k-1}+2^{(k+1)-1}=2^k-1+2^{(k+1)-1}$

$$1+2+2^2+\cdots+2^{k-1}+2^{(k+1)-1}=2^k-1+2^k$$
$$1+2+2^2+\cdots+2^{k-1}+2^{(k+1)-1}=2\cdot 2^k-1$$
$$1+2+2^2+\cdots+2^{k-1}+2^{(k+1)-1}=2^{k+1}-1$$

Thus, $1+2+2^2+\cdots+2^{n-1}=2^n-1$ is true for all positive integers *n*. *(10.4 #17)*

Section 10.5
The Binomial Theorem

Who Knew That First?

Telephones, Internet, and other modern forms of communication mean that information now can spread across the globe in the blink of an eye.

In this section, we study a special array of numbers known as Pascal's triangle, named after mathematician Blaise Pascal.
However, this triangular array of numbers actually appeared centuries earlier in a Chinese document.

The same mathematics is often discovered by independent researchers separated by time, place, and culture.
But with modern communication, important discoveries are now shared much more efficiently.

Objective #1: Evaluate a binomial coefficient.

✔ *Solved Problem #1*

1a. Evaluate: $\binom{6}{3}$.

$$\binom{6}{3} = \frac{6!}{3!(6-3)!}$$
$$= \frac{6!}{3!3!}$$
$$= \frac{6\cdot5\cdot4\cdot\cancel{3!}}{3\cdot2\cdot1\cdot\cancel{3!}}$$
$$= 20$$

1b. Evaluate: $\binom{6}{0}$.

$$\binom{6}{0} = \frac{6!}{0!(6-0)!}$$
$$= \frac{6!}{6!}$$
$$= 1$$

✎ *Pencil Problem #1* ✎

1a. Evaluate: $\binom{8}{3}$.

1b. Evaluate: $\binom{12}{1}$.

1c. Evaluate: $\binom{8}{2}$.

$$\binom{8}{2} = \frac{8!}{2!(8-2)!}$$
$$= \frac{8!}{2!6!}$$
$$= \frac{8 \cdot 7}{2}$$
$$= 28$$

1c. Evaluate: $\binom{100}{2}$.

1d. Evaluate: $\binom{3}{3}$.

$$\binom{3}{3} = \frac{3!}{3!(3-3)!}$$
$$= \frac{3!}{3!0!}$$
$$= \frac{3!}{3!}$$
$$= 1$$

1d. Evaluate: $\binom{6}{6}$.

Objective #2: Expand a binomial raised to a power.

✔ *Solved Problem #2a*

2a. Expand: $(x+1)^4$

$$(x+1)^4$$
$$= \binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{3}x + \binom{4}{4}$$
$$= x^4 + 4x^3 + 6x^2 + 4x + 1$$

✎ *Pencil Problem #2a* ✐

2a. Expand: $(x+2)^3$

✔ *Solved Problem #2b*

2b. Expand: $(x-2y)^5$

$$(x-2y)^5 = \binom{5}{0}x^5(-2y)^0 + \binom{5}{1}x^4(-2y)^1 + \binom{5}{2}x^3(-2y)^2 + \binom{5}{3}x^2(-2y)^3 + \binom{5}{4}x(-2y)^4 + \binom{5}{5}x^0(-2y)^5$$

$$= x^5 \quad -5x^4(2y) \quad +10x^3(4y^2) \quad -10x^2(8y^3) \quad +5x(16y^4) \quad -32y^5$$

$$= x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$$

✎ *Pencil Problem #2b* ✐

2b. Expand: $\left(x^2+2y\right)^4$

Objective #3: Find a particular term in a binomial expansion.

✔ **Solved Problem #3**

3. Find the fifth term in the expansion of $(2x+y)^9$.

Since we are looking for the 5th term, $r = 5-1 = 4$.
Thus, $r = 4$, $a = 2x$, $b = y$, and $n = 9$.

$$(r+1)\text{st term } = \binom{n}{r} a^{n-r} b^r$$
$$\text{fifth term } = \binom{9}{4}(2x)^5 y^4$$
$$= \frac{9!}{4!5!}(32x^5)y^4$$
$$= 4032x^5y^4$$

✎ **Pencil Problem #3** ✐

3. Find the sixth term in the expansion of $\left(x^2 + y^3\right)^8$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 56 *(10.5 #1)*

1b. 12 *(10.5 #3)*

1c. 4950 *(10.5 #7)*

1d. 1 *(10.5 #5)*

2a. $x^3 + 6x^2 + 12x + 8$ *(10.5 #9)*

2b. $x^8 + 8x^6y + 24x^4y^2 + 32x^2y^3 + 16y^4$ *(10.5 #17)*

3. $56x^6y^{15}$ *(10.5 #43)*

Section 10.6
Counting Principles, Permutations, and Combinations

I Have NOTHING to Wear!

On many mornings, we feel quite limited on the fashion statement we desire to make that day. But the truth is we usually have more clothing options than we might think. If we consider how the various components of our outfit can be mixed and matched, the number of unique outfits can be difficult to count.

Attempting to count each possibility one-by-one can be daunting in many situations.

In this section of the textbook, we will use organized mathematical methods and formulas that will allow us to count more quickly and accurately than the 1, 2, 3, 4, 5, 6, … method.

Objective #1: Use the Fundamental Counting Principle.

✔ *Solved Problem #1*

1a. A pizza can be ordered with three choices of size (small, medium, or large), four choices of crust (thin, thick, crispy, or regular), and six choices of toppings (ground beef, sausage, pepperoni, bacon, mushrooms, or onions). How many different one-topping pizzas can be ordered?

Multiply the number of choices for each of the three decisions:

$$\underline{\text{Size}}: \quad \underline{\text{Crust}}: \quad \underline{\text{Topping}}:$$
$$3 \times 4 \times 6 = 72$$

72 different one-topping pizzas can be ordered.

1b. License plates in a particular state display two letters followed by three numbers, such as AT-887 or BB-013. How many different license plates can be manufactured?

Multiply the number of choices for each of the letters and each of the digits:

$$\overbrace{26}^{\text{Letter 1}} \times \overbrace{26}^{\text{Letter 2}} \times \overbrace{10}^{\text{Digit 1}} \times \overbrace{10}^{\text{Digit 2}} \times \overbrace{10}^{\text{Digit 3}} = 676{,}000$$

676,000 different license plates can be manufactured.

✎ *Pencil Problem #1* ✐

1a. An ice cream store sells two drinks (sodas or milk shakes), in four sizes (small, medium, large, or jumbo), and five flavors (vanilla, strawberry, chocolate, coffee, or pistachio). In how many ways can a customer order a drink?

1b. You are taking a multiple-choice test that has five questions. Each of the questions has three answer choices, with one correct answer per question. If you select one of these three choices for each question and leave nothing blank, in how many ways can you answer the questions?

Objective #2: Use the permutations formula.

✔ *Solved Problem #2*

2a. A corporation has seven members on its board of directors. In how many different ways can it elect a president, vice-president, secretary, and treasurer?

The corporation is choosing 4 officers from a group of 7 people. The order in which the officers are chosen matters because the president, vice-president, secretary, and treasurer each have different responsibilities. Thus, we are looking for the number of permutations of 7 things taken 4 at a time.

$${}_7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 840$$

There are 840 ways of filling the four offices.

2b. In how many ways can 6 books be lined up along a shelf?

Because you are using all six of your books in every possible arrangement, you are arranging 6 books from a group of 6 books. Thus, we are looking for the number of permutations of 6 things taken 6 at a time.

$${}_6P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1} = 720$$

There are 720 ways the 6 books can be lined up along the shelf.

✎ *Pencil Problem #2*

2a. Using 15 flavors of ice cream, how many cones with three different flavors can you create if it is important to you which flavor goes on the top, middle, and bottom?

2b. What is the number of permutations of 8 things taken 0 at a time?

Objective #3: Distinguish between permutation problems and combination problems.

✔ *Solved Problem #3*

3a. Determine if the question involves combinations or permutations. (Do *not* solve the problem.)

How many ways can you select 6 free DVDs from a list of 200 DVDs?

The order in which the DVDs are selected does not matter.

Thus, this problem involves combinations.

✎ *Pencil Problem #3*

3a. Determine if the question involves combinations or permutations. (Do *not* solve the problem.)

A medical researcher needs 6 people to test the effectiveness of an experimental drug. If 13 people have volunteered for the test, in how many ways can 6 people be selected?

3b. Determine if the question involves combinations or permutations. (Do *not* solve the problem.)
In a race in which there are 50 runners and no ties, in how many ways can the first three finishers come in?

The order in which the runners finish does matter.

Thus, this problem involves permutations.

3b. Determine if the question involves combinations or permutations. (Do *not* solve the problem.)
How many different four-letter passwords can be formed from the letters A, B, C, D, E, F, and G if no repetition of letters is allowed?

Objective #4: Use the combinations formula.

✔ *Solved Problem #4*

4a. From a group of 10 physicians, in how many ways can four people be selected to attend a conference on acupuncture?

The order in which the four people are selected does not matter. This is a problem of selecting 4 people from a group of 10 people. We are looking for the number of combinations of 10 things taken 4 at a time.

$$
\begin{aligned}
{}_{10}C_4 &= \frac{10!}{(10-4)!4!} \\
&= \frac{10!}{6!4!} \\
&= \frac{10\cdot 9\cdot 8\cdot 7\cdot 6!}{6!\cdot 4\cdot 3\cdot 2\cdot 1} \\
&= 210
\end{aligned}
$$

The four attendees can be selected in 210 different ways.

✎ *Pencil Problem #4* ✎

4a. An election ballot asks voters to select three city commissioners from a group of six candidates. In how many ways can this be done?

4b. How many different 4-card hands can be dealt from a deck that has 16 different cards?

Because the order in which the 4 cards are dealt does not matter, this is a problem involving combinations. We are looking for the number of combinations of 16 cards drawn 4 at a time.

$$\begin{aligned} {}_{16}C_4 &= \frac{16!}{(16-4)!4!} \\ &= \frac{16!}{12!4!} \\ &= \frac{16\cdot 15\cdot 14\cdot 13\cdot 12!}{12!\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{16\cdot 15\cdot 14\cdot 13\cdot \cancel{12!}}{\cancel{12!}\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{16\cdot 15\cdot 14\cdot 13}{4\cdot 3\cdot 2\cdot 1} \\ &= 1820 \end{aligned}$$

There are 1820 different 4-card hands.

4b. You volunteer to help drive children at a charity event to the zoo, but you can fit only 8 of the 17 children present in your van. How many different groups of 8 children can you drive?

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 40 *(10.6 #31)* **1b.** 243 *(10.6 #33)*

2a. 2730 *(10.6 #65)* **2b.** 1 *(10.6 #7)*

3a. combinations *(10.6 #17)* **3b.** permutations *(10.6 #19)*

4a. 20 *(10.6 #49)* **4b.** 24,310 *(10.6 #53)*

Section 10.7
Probability

The Weather Outside is Frightful!

Have you ever thought about the chances of being
hit by lightning, caught in a tornado, hurricane,
or some other major weather event?

In one of the application exercises in this section, mathematicians, meteorologists, and you will team up to determine such probabilities.

Objective #1: Compute empirical probability.

✔ *Solved Problem #1*

1. Use the data in the table to find the probabilities.

Mammography Screening on 100,000 U.S. Women, Ages 40 to 50	Breast Cancer	No Breast Cancer
Positive Mammogram	720	6944
Negative Mammogram	80	92,256

1a. Find the probability that a woman aged 40 to 50 has a positive mammogram.

The probability of having a positive mammogram is the number of women with a positive mammogram divided by the total number of women.

$$\begin{aligned} P(\text{positive mammogram}) &= \frac{720+6944}{100{,}000} \\ &= \frac{7664}{100{,}000} \\ &\approx 0.077 \end{aligned}$$

✎ *Pencil Problem #1* ✎

1. The table shows the distribution, by marital status and gender, of the 242 million Americans ages 18 or older. Use the table to find the probabilities.

	Never Married	Married	Widowed	Divorced
Male	40	65	3	10
Female	34	65	11	14

1a. If one person is randomly selected from the population described in the table, find the probability, to the nearest hundredth, that the person is divorced.

1b. Among women with positive mammograms, find the probability of having breast cancer.
(Use the data in the table on the previous page)

To find the probability of breast cancer among women with positive mammograms, restrict the data to women with positive mammograms:

Mammography Screening on 100,000 U.S. Women, Ages 40 to 50	Breast Cancer	No Breast Cancer
Positive Mammogram	720	6944

$$P(\text{breast cancer}) = \frac{720}{720+6944} = \frac{720}{7664} \approx 0.094$$

1b. Among those who are divorced, find the probability of selecting a woman.
(Use the data in the table on the previous page)

Objective #2: Compute theoretical probability.

✔ *Solved Problem #2*

2a. A die is rolled. Find the probability of getting a number greater than 4.

Two of the six numbers, 5 and 6, are greater than 4.

$$P(\text{greater than 4}) = \frac{2}{6} = \frac{1}{3}$$

2b. The original Florida LOTTO was set up so that each player chose six different numbers from 1 to 49. With one LOTTO ticket, what was the probability of winning the top cash prize? Express the answer as a fraction and as a decimal correct to ten places.

$$_{49}C_6 = \frac{49!}{(49-6)!6!} = \frac{49!}{43!6!} = \frac{49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44\cdot \cancel{43!}}{\cancel{43!}\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13{,}983{,}816$$

$$P(\text{winning LOTTO}) = \frac{1}{13{,}983{,}816} \approx 0.0000000715$$

✎ *Pencil Problem #2* ✎

2a. A die is rolled. Find the probability of getting a 4.

2b. To play the California lottery, a person has to correctly select 6 out of 51 numbers. If you pick six numbers that are the same as the ones drawn by the lottery, you win. What is the probability that a person with one combination of six numbers will win?

Objective #3: Find the probability that an event will not occur.

✔ *Solved Problem #3*

3. Of the 7000 million people in the world, 550 million live in North America. If one person is randomly selected from the world population, find the probability that the person does not live in North America.

$$\begin{aligned} P(\text{not North America}) &= 1 - P(\text{North America}) \\ &= 1 - \frac{550}{7000} \\ &= \frac{6450}{7000} \\ &= \frac{129}{140} \end{aligned}$$

Pencil Problem #3

3. If you are dealt one card from a 52-card deck, find the probability that you are *not* dealt a king.

Objective #4: Find the probability of one event or a second event occurring.

✔ *Solved Problem #4*

4a. If you roll a single, six-sided die, what is the probability of getting either a 4 or a 5?

These events are mutually exclusive.
Thus, add their individual probabilities.

$$\begin{aligned} P(4 \text{ or } 5) &= P(4) + P(5) \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

4b. Each number, 1 through 8, is written on slips of paper and placed in a hat. If one number is selected at random, find the probability that the number selected will be an odd number or a number less than 5.

These events are *not* mutually exclusive. Thus, use the formula $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

$$\begin{aligned} &P(\text{odd or less than 5}) \\ &= P(\text{odd}) + P(\text{less than 5}) - P(\text{odd and less than 5}) \\ &= \frac{4}{8} + \frac{4}{8} - \frac{2}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

Pencil Problem #4

4a. If you are dealt one card from a 52-card deck, find the probability that you are dealt a 2 or a 3.

4b. Each number, 1 through 8, is written on slips of paper and placed in a hat. If one number is selected at random, find the probability that the number selected will be an odd number or a number less than 6.

Objective #5: Find the probability of one event and a second event occurring.

✔ *Solved Problem #5*

5a. On a roulette wheel, the ball can land with equal probability on any one of the 38 numbered slots, two of which are green. Find the probability of green occurring on two consecutive plays.

The events are independent.
Thus, use the formula $P(A \text{ and } B) = P(A) \cdot P(B)$.

$$\begin{aligned} P(\text{green and green}) &= P(\text{green}) \cdot P(\text{green}) \\ &= \frac{2}{38} \cdot \frac{2}{38} \\ &= \frac{1}{361} \\ &\approx 0.00277 \end{aligned}$$

5b. Find the probability of a family having four boys in a row.

The events are independent.
Thus, multiply their probabilities.

$$\begin{aligned} P(\text{4 boys in a row}) &= P(\text{boy and boy and boy and boy}) \\ &= P(\text{boy}) \cdot P(\text{boy}) \cdot P(\text{boy}) \cdot P(\text{boy}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{16} \end{aligned}$$

✎ *Pencil Problem #5* ✐

5a. A single die is rolled twice. Find the probability of rolling a 2 the first time and a 3 the second time.

5b. If you toss a fair coin six times, what is the probability of getting all heads?

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 0.10 *(10.7 #1)* **1b.** 0.58 *(10.7 #7)* **2a.** $\frac{1}{6}$ *(10.7 #11)* **2b.** $\frac{1}{18{,}009{,}460} \approx 0.0000000555$ *(10.7 #27)*

3. $\frac{12}{13}$ *(10.7 #37)* **4a.** $\frac{2}{13}$ *(10.7 #39)* **4b.** $\frac{3}{4}$ *(10.7 #43)* **5a.** $\frac{1}{36}$ *(10.7 #47)* **5b.** $\frac{1}{64}$ *(10.7 #51)*