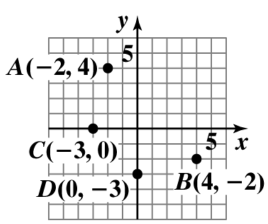
Section 1.1
Graphs and Graphing Utilities

Let it snow! Let it snow! Let it snow!

The arrival of snow can range from light flurries to a full-fledged blizzard. Snow can be welcomed as a beautiful backdrop to outdoor activities or it can be a nuisance and endanger drivers.

We will look at how graphs can be used to explain both mathematical concepts and everyday situations. Specifically, in the application exercises of this section of the textbook, you will match stories of varying snowfalls to the graphs that explain them.

Objective #1: Plot points in the rectangular coordinate system.

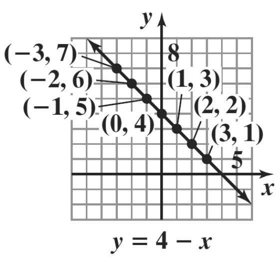
✔ *Solved Problem #1*	✎ *Pencil Problem #1* ✎
1a. Plot the points: $A(-2, 4)$, $B(4, -2)$, $C(-3, 0)$, and $D(0, -3)$. From the origin, point A is left 2 units and up 4 units. From the origin, point B is right 4 units and down 2 units. From the origin, point C is left 3 units. From the origin, point D is down 3 units. 	**1a.** Plot the points: $A(1, 4)$, $B(-2, 3)$, $C(-3, -5)$, and $D(-4, 0)$.
1b. If a point is on the x-axis it is neither up nor down, so $x = 0$. False. The y-coordinate gives the distance up or down, so $y = 0$.	**1b.** True or false: If a point is on the y-axis, its x-coordinate must be 0.

Objective #2: Graph equations in the rectangular coordinate system.

✔ Solved Problem #2

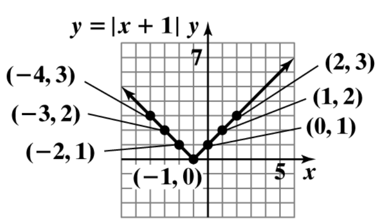
2a. Graph $y = 4 - x$.

x	$y = 4 - x$	(x, y)
-3	$y = 4-(-3) = 7$	$(-3,7)$
-2	$y = 4-(-2) = 6$	$(-2,6)$
-1	$y = 4-(-1) = 5$	$(-1,5)$
0	$y = 4-(0) = 4$	$(0,4)$
1	$y = 4-(1) = 3$	$(1,3)$
2	$y = 4-(2) = 2$	$(2,2)$
3	$y = 4-(3) = 1$	$(3,1)$



2b. Graph $y = |x+1|$.

x	$y = \|x+1\|$	(x, y)
-4	$y = \|-4+1\| = \|-3\| = 3$	$(-4,3)$
-3	$y = \|-3+1\| = \|-2\| = 2$	$(-3,2)$
-2	$y = \|-2+1\| = \|-1\| = 1$	$(-2,1)$
-1	$y = \|-1+1\| = \|0\| = 0$	$(-1,0)$
0	$y = \|0+1\| = \|1\| = 1$	$(0,1)$
1	$y = \|1+1\| = \|2\| = 2$	$(1,2)$
2	$y = \|2+1\| = \|3\| = 3$	$(2,3)$



✎ Pencil Problem #2 ✎

2a. Graph $y = x^2 - 2$. Let $x = -3, -2, -1, 0, 1, 2,$ and 3.

2b. Graph $y = 2|x|$. Let $x = -3, -2, -1, 0, 1, 2,$ and 3.

Objective #3: Interpret information about a graphing utility's viewing rectangle or table.

✔ *Solved Problem #3*

3. What is the meaning of a [–100, 100, 50] by [–100, 100, 10] viewing rectangle?

The minimum x-value is –100, the maximum x-value is 100, and the distance between consecutive tick marks is 50.

The minimum y-value is –100, the maximum y-value is 100, and the distance between consecutive tick marks is 10.

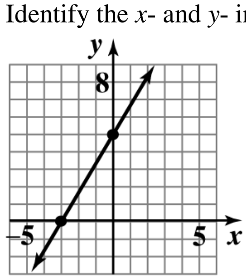
✎ *Pencil Problem #3* ✐

3. What is the meaning of a [–20, 80, 10] by [–30, 70, 10] viewing rectangle?

Objective #4: Use a graph to identify intercepts.

✔ *Solved Problem #4*

4a. Identify the x- and y- intercepts:

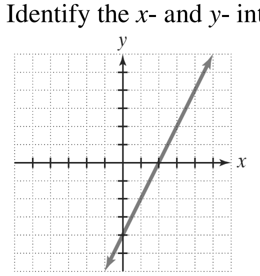


The graph crosses the x-axis at (–3,0). Thus, the x-intercept is –3.

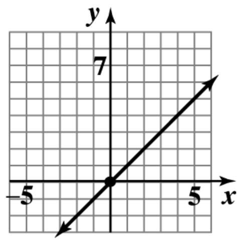
The graph crosses the y-axis at (0,5). Thus, the y-intercept is 5.

✎ *Pencil Problem #4* ✐

4a. Identify the x- and y- intercepts:



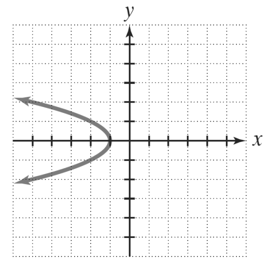
4b. Identify the x- and y- intercepts:



The graph crosses the x-axis at (0,0).
Thus, the x-intercept is 0.

The graph crosses the y-axis at (0,0).
Thus, the y-intercept is 0.

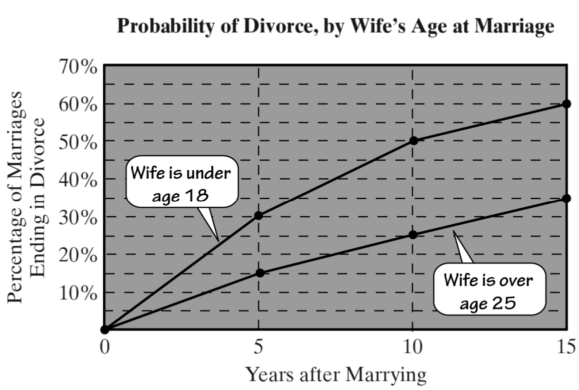
4b. Identify the x- and y- intercepts:



Objective #5: Interpret information given by graphs.

✔ *Solved Problem #5*

5. The line graphs show the percentage of marriages ending in divorce based on the wife's age at marriage.



The model $d = 4n + 5$ approximates the data in the graph when the wife is under 18 at the time of marriage. In the model, n is the number of years after marriage and d is the percentage of marriages ending in divorce.

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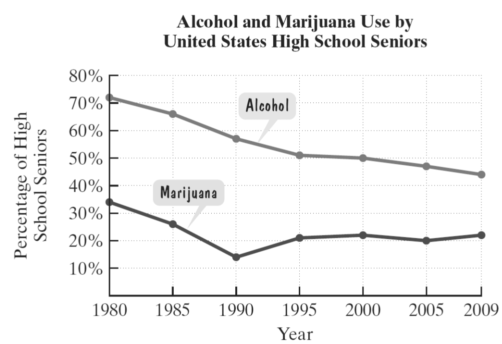
✎ *Pencil Problem #5* ✎

5. The graphs show the percentage of high school seniors who used alcohol or marijuana.



Source: U.S. Department of Health and Human Services

The data for seniors who used marijuana can be modeled by $M = -0.3n + 27$, where M is the percentage of seniors who used marijuana n years after 1980.

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5a. Use the formula to determine the percentage of marriages ending in divorce after 15 years when the wife is under 18 at the time of marriage.

$d = 4n + 5$

$d = 4(15) + 5$

$d = 60 + 5$

$d = 65$

According to the formula, 65% of marriages end in divorce after 15 years when the wife is under 18 at the time of marriage.

5a. Use the formula to determine the percentage of seniors who used marijuana in 2005.

5b. Use the appropriate line graph to determine the percentage of marriages ending in divorce after 15 years when the wife is under 18 at the time of marriage.

Locate 15 on the horizontal axis and locate the point above it on the graph. Read across to the corresponding percentage on the vertical axis. This percentage is 60. According to the line graph, 60% of marriages end in divorce after 15 years when the wife is under 18 at the time of marriage.

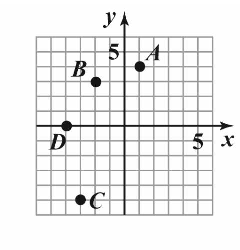
5b. Use the appropriate line graph to determine the percentage of seniors who used marijuana in 2005.

5c. Does the value given by the model underestimate or overestimate the value shown by the graph? By how much?

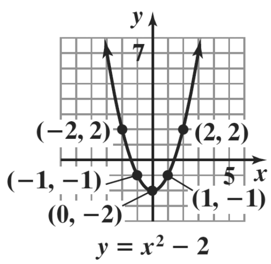
The value given by the model, 65%, is greater than the value shown by the graph, 60%, so the model overestimates the percentage by 65 – 5, or 5.

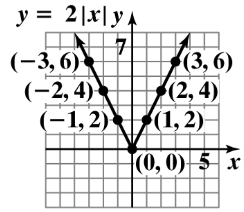
5c. Does the formula underestimate or overestimate the percentage of seniors who used marijuana in 2005 as shown by the graph.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 

(1.1 #1-9) **1b.** True *(1.1.#73)*

2a. 

(1.1 #13) **2b.** 

(1.1 #21)

3. The minimum x-value is –20, the maximum x-value is 80, and the distance between consecutive tick marks is 10. The minimum y-value is –30, the maximum y-value is 70, and the distance between consecutive tick marks is 10. *(1.1 #31)*

4a. x-intercept: 2; y-intercept: -4 *(1.1 #41)* **4b.** x-intercept: -1 ; y-intercept: none *(1.1 #45)*

5a. 19.5% *(1.1 #55b)* **5b.** 20% *(1.1 #55a)* **5c.** underestimates by 0.5 *(1.1 #55b)*

Section 1.2
Basics of Functions and Their Graphs

Say *WHAT???*

You may have noticed that mathematical notation occasionally can have more than one meaning depending on the context.

For example, $(-3,6)$ could refer to the ordered pair where $x=-3$ and $y=6$, or it could refer to the open interval $-3<x<6$.

Similarly, in this section of the textbook, we will use the notation, $f(x)$. It may surprise you to find out that it does *not* mean to multiply "f times x."

It will be important for you to gain an understanding of what this notation *does* mean as you work through this essential concept of "functions."

Objective #1: Find the domain and range of a relation.

✔ *Solved Problem #1*

1. Find the domain and range of the relation:
{(0, 9.1), (10, 6.7), (20, 10.7), (30, 13.2), (40, 21.2)}

The domain is the set of all first components.

Domain:
{0, 10, 20, 30, 40}.

The range is the set of all second components.

Range:
{9.1, 6.7, 10.7, 13.2, 21.2}.

✎ *Pencil Problem #1* ✎

1. Find the domain and range of the relation:
{(3, 4), (3, 5), (4, 4), (4, 5)}

Objective #2: Determine whether a relation is a function.

✔ *Solved Problem #2*

2a. Determine whether the relation is a function:
$\{(1,2),(3,4),(5,6),(5,8)\}$

5 corresponds to both 6 and 8. If any element in the domain corresponds to more than one element in the range, the relation is not a function.

Thus, the relation is not a function.

✎ *Pencil Problem #2* ✎

2a. Determine whether the relation is a function:
{(3, 4), (3, 5), (4, 4), (4, 5)}

2b. Determine whether the relation is a function:
$$\{(1,2),(3,4),(6,5),(8,5)\}$$

Every element in the domain corresponds to exactly one element in the range. No two ordered pairs in the given relation have the same first component and different second components.

Thus, the relation is a function.

2b. Determine whether the relation is a function:
$$\{(-3,-3),(-2,-2),(-1,-1),(0,0)\}$$

Objective #3: Determine whether an equation represents a function.

✔ *Solved Problem #3*

3. Solve each equation for *y* and then determine whether the equation defines *y* as a function of *x*.

3a. $2x + y = 6$

Subtract 2*x* from both sides to solve for *y*.

$$2x + y = 6$$
$$2x - 2x + y = 6 - 2x$$
$$y = 6 - 2x$$

For each value of *x*, there is only one value of *y*, so the equation defines *y* as a function of *x*.

3b. $x^2 + y^2 = 1$

Subtract x^2 from both sides and then use the square root property to solve for *y*.

$$x^2 + y^2 = 1$$
$$x^2 - x^2 + y^2 = 1 - x^2$$
$$y^2 = 1 - x^2$$
$$y = \pm\sqrt{1 - x^2}$$

For values of *x* between −1 and 1, there are two values of *y*. For example, if $x = 0$, then $y = \pm 1$. Thus, the equation does not define *y* as a function of *x*.

✎ *Pencil Problem #3* ✐

3. Solve each equation for *y* and then determine whether the equation defines *y* as a function of *x*.

3a. $x^2 + y = 16$

3b. $x = y^2$

Objective #4: Evaluate a function.

✔ *Solved Problem #4*

4. If $f(x) = x^2 - 2x + 7,$ evaluate each of the following.

4a. $f(-5)$

Substitute −5 for *x*. Place parentheses around −5 when making the substitution.

$$f(-5) = (-5)^2 - 2(-5) + 7$$
$$= 25 + 10 + 7 = 42$$

4b. $f(x + 4)$

Substitute $x + 4$ for x and then simplify. Place parentheses around $x + 4$ when making the substitution.

Use $(A+B)^2 = A^2 + 2AB + B^2$ to expand $(x+4)^2$ and the distributive property to multiply $-2(x + 4)$. Then combine like terms.

$$f(x+4) = (x+4)^2 - 2(x+4) + 7$$
$$= x^2 + 8x + 16 - 2x - 8 + 7$$
$$= x^2 + 6x + 15$$

4c. $f(-x)$

Substitute $-x$ for x. Place parentheses around $-x$ when making the substitution.

$$f(-x) = (-x)^2 - 2(-x) + 7$$
$$= x^2 + 2x + 7$$

✎ *Pencil Problem #4* ✐

4. If $g(x) = x^2 + 2x + 3,$ evaluate each of the following.

4a. $g(-1)$

4b. $g(x + 5)$

4c. $g(-x)$

Objective #5: Graph functions by plotting points.

✔ *Solved Problem #5*

5. Graph the functions $f(x) = 2x$ and $g(x) = 2x - 3$ in the same rectangular coordinate system. Select integers for *x*, starting with -2 and ending with 2. How is the graph of *g* related to the graph of *f* ?

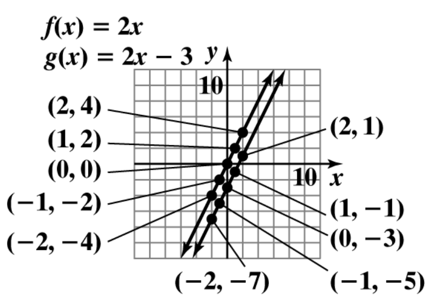
Make a table for $f(x) = 2x$:

x	$f(x) = 2x$	(x, y)
-2	$f(-2) = 2(-2) = -4$	$(-2, -4)$
-1	$f(-1) = 2(-1) = -2$	$(-1, -2)$
0	$f(0) = 2(0) = 0$	$(0, 0)$
1	$f(1) = 2(1) = 2$	$(1, 2)$
2	$f(2) = 2(2) = 4$	$(2, 4)$

Make a table for $g(x) = 2x - 3$:

x	$g(x) = 2x - 3$	(x, y)
-2	$g(-2) = 2(-2) - 3 = -7$	$(-2, -7)$
-1	$g(-1) = 2(-1) - 3 = -5$	$(-1, -5)$
0	$g(0) = 2(0) - 3 = -3$	$(0, -3)$
1	$g(1) = 2(1) - 3 = -1$	$(1, -1)$
2	$g(2) = 2(2) - 3 = 1$	$(2, 1)$

Plot the points and draw the lines that pass through them.



The graph of *g* is the graph of *f* shifted down by 3 units.

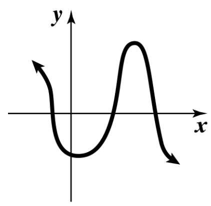
✎ *Pencil Problem #5* ✎

5. Graph the functions $f(x) = |x|$ and $g(x) = |x| - 2$ in the same rectangular coordinate system. Select integers for *x*, starting with -2 and ending with 2. How is the graph of *g* related to the graph of *f* ?

Objective #6: Use the vertical line test to identify functions.

✔ *Solved Problem #6*

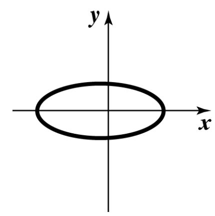
6. Use the vertical line test to determine if the graph represents y as a function of x.



The graph passes the vertical line test and thus y is a function of x.

✎ *Pencil Problem #6* ✎

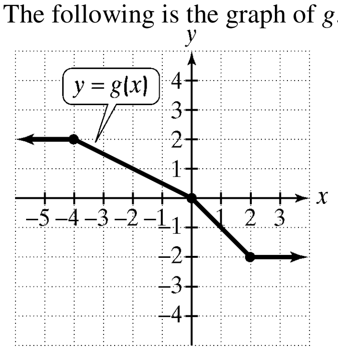
6. Use the vertical line test to determine if the graph represents y as a function of x.



Objective #7: Obtain information about a function from its graph.

✔ *Solved Problem #7*

7a. The following is the graph of g.



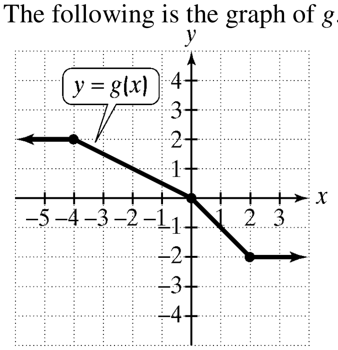
Use the graph to find $g(-20)$.

The graph indicates that to the left of $x = -4$, the graph is at a constant height of 2.

Thus, $g(-20) = 2$.

✎ *Pencil Problem #7* ✎

7a. The following is the graph of g.



Use the graph to find $g(-4)$.

7b. Use the graph from *Problem 7a* above to find the value of x for which $g(x) = -1$.

$g(1) = -1$

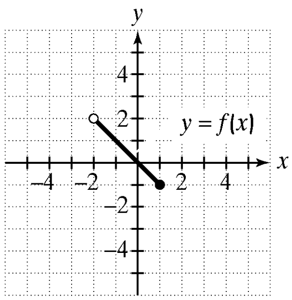
The height of the graph is –1 when $x = 1$.

7b. Use the graph from *Problem 7a* above to find the value of x for which $g(x) = 1$.

Objective #8: Identify the domain and range of a function from its graph.

✔ *Solved Problem #8*

8. Use the graph of the function to identify its domain and its range.

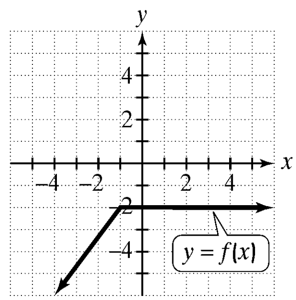


Inputs on the x-axis extend from –2, excluding −2, to 1, including 1.
The domain is $(-2,1]$.

Outputs on the y-axis extend from –1, including −1, to 2, excluding 2.
The range is $[-1,2)$.

✎ *Pencil Problem #8* ✎

8. Use the graph of the function to identify its domain and its range.



Objective #9: Identify intercepts from a function's graph.

✔ *Solved Problem #9*

9. True or false: The graph of a function may cross the y-axis several times, so the graph may have more than one y-intercept.

False. Since each point on the y-axis has x-coordinate 0 and a function may have only one y-value for each x-value, the graph of a function has at most one y-coordinate.

✎ *Pencil Problem #9* ✎

9. True or false: The graph of a function may cross the x-axis several times, so the graph may have more than one x-intercept.

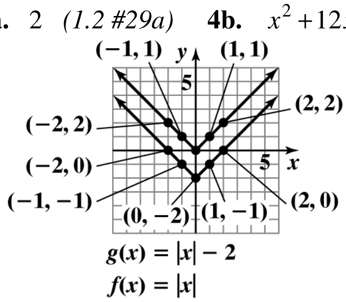
<u>**Answers**</u> for Pencil Problems <u>*(Textbook Exercise references in parentheses)*</u>:

1. Domain: {3, 4}. Range: {4, 5}. *(1.2 #3)* **2a.** not a function *(1.2 #3)* **2b.** function *(1.2 #7)*

3a. $y = 16 - x^2$; y is a function of x. *(1.2 #13)* **3b.** $y = \pm\sqrt{x}$; y is not a function of x. *(1.2 #17)*

4a. 2 *(1.2 #29a)* **4b.** $x^2 + 12x + 38$ *(1.2 #29b)* **4c.** $x^2 - 2x + 3$ *(1.2 #29c)*

5.



The graph of g is the graph of f shifted down 2 units. *(1.2 #45)*

6. not a function *(1.2 #59)* **7a.** 2 *(1.2 #71)* **7b.** −2 *(1.2 #75)*

8. Domain: $(-\infty,\infty)$. Range: $(-\infty,-2]$ *(1.2 #87)* **9.** True *(1.2 #77)*

Section 1.3
More on Functions and Their Graphs

Can I Really Tell That from a Graph?

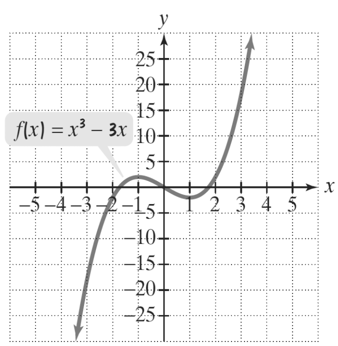
Graphs provide a visual representation of how a function changes over time. Many characteristics of a function are much more evident from the function's graph than from the equation that defines the function.

We can use graphs to determine for what years the fuel efficiency of cars was increasing and decreasing and at what age men and women attain their maximum percent body fat. A graph can even help you understand your cellphone bill better.

Objective #1: Identify intervals on which a function increases, decreases, or is constant.

✔ *Solved Problem #1*

1. State the intervals on which the given function is increasing, decreasing, or constant.



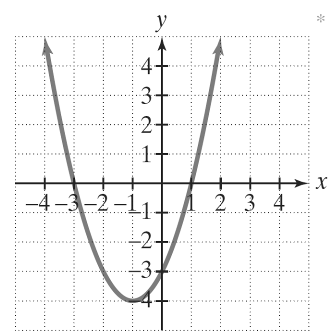
The intervals are stated in terms of x-values.
When we start at the left and follow along the graph, at first the graph is going up. This continues until $x = -1$. The function is increasing on the interval $(-\infty, -1)$.

At $x = -1$, the graph turns and moves downward until we get to $x = 1$. The function is decreasing on the interval $(-1, 1)$.

At $x = 1$, the graph turns again and continues in an upward direction. The function is increasing on the interval $(1, \infty)$.

✎ *Pencil Problem #1* ✎

1. State the intervals on which the given function is increasing, decreasing, or constant.



Objective #2: Use graphs to locate relative maxima or minima.

✔ *Solved Problem #2*

2. Look at the graph in Solved Problem #1. Locate values at which the function f has any relative maxima or minima. What are these relative maxima or minima?

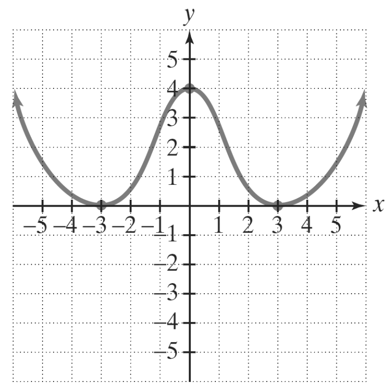
The graph has a turning point at $x = -1$. The value of $f(x)$ or y at $x = -1$ is greater than the values of $f(x)$ for values of x near -1 (for values of x between -2 and 0, for example). Thus, f has a relative maximum at $x = -1$. The relative maximum is the value of $f(x)$ or y corresponding to $x = -1$. Using the equation in the graph, we find that $f(-1) = (-1)^3 - 3(-1) = 2$. We say that f has a relative maximum of 2 at $x = -1$.

The graph has a second turning point at $x = 1$. The value of $f(x)$ or y at $x = 1$ is less than the values of $f(x)$ for values of x near 1 (for values of x between 0 and 2, for example). Thus, f has a relative minimum at $x = 1$. The relative minimum is the value of $f(x)$ or y corresponding to $x = 1$. Using the equation in the graph, we find that $f(1) = (1)^3 - 3(1) = -2$. We say that f has a relative minimum of -2 at $x = 1$.

Note that the relative maximum occurs where the functions changes from increasing to decreasing and the relative minimum occurs where the graph changes from decreasing to increasing.

✎ *Pencil Problem #2* ✐

2. The graph of a function f is given below. Locate values at which the function f has any relative maxima or minima. What are these relative maxima or minima? Read y-values from the graph, as needed, since the equation is not given.



Objective #3: Identify even or odd functions and recognize their symmetries.

✔ *Solved Problem #3*

3. Determine whether each of the following functions is even, odd, or neither.

3a. $f(x) = x^2 + 6$

Replace x with $-x$.

$f(-x) = (-x)^2 + 6 = x^2 + 6 = f(x)$

The function did not change when we replaced x with $-x$. The function is even.

✎ *Pencil Problem #3* ✐

3. Determine whether each of the following functions is even, odd, or neither.

3a. $f(x) = x^3 + x$

3b. $g(x) = 7x^3 - x$

Replace x with $-x$.
$g(-x) = 7(-x)^3 - (-x) = -7x^3 + x = -g(x)$

Each term of the equation defining the function changed sign when we replaced x with $-x$. The function is odd.

3b. $g(x) = x^2 + x$

3c. $h(x) = x^5 + 1$

Replace x with $-x$.
$h(x) = (-x)^5 + 1 = -x^5 + 1$

The resulting function is not equal to the original function, so the function is not even. Only the sign of one term changed, so the function is not odd. The function is neither even nor odd.

3c. $h(x) = x^2 - x^4$

Objective #4: Understand and use piecewise functions.

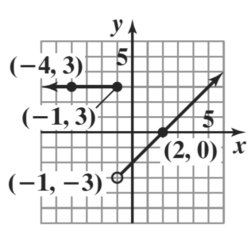
✔ *Solved Problem #4*

4. Graph the piecewise function defined by

$$f(x) = \begin{cases} 3 & \text{if} \quad x \le -1 \\ x - 2 & \text{if} \quad x > -1 \end{cases}$$

For $x \le -1$, the function value is always 3, so (–4, 3) and (–1, 3) are examples of points on the first piece of the graph.

For $x > -1$, we use $f(x) = x - 2$. We have points such as (0, –2) and (2, 0) on the graph. Note that this piece of the graph will approach the point (–1, –3) but this point is not part of the graph. We use an open dot at (–1, –3).



$$f(x) = \begin{cases} 3 & \text{if } x \le -1 \\ x - 2 & \text{if } x > -1 \end{cases}$$

✎ *Pencil Problem #4* ✎

4. Graph the piecewise function defined by

$$f(x) = \begin{cases} 2x & \text{if} \quad x \le 0 \\ 2 & \text{if} \quad x > 0 \end{cases}$$

Objective #5: Find and simplify a function's difference quotient.

✔ *Solved Problem #5*

5. Find and simplify the difference quotient for $f(x) = -2x^2 + x + 5$.

$$\frac{f(x+h) - f(x)}{h}$$
$$= \frac{[-2(x+h)^2 + (x+h) + 5] - (-2x^2 + x + 5)}{h}$$
$$= \frac{-2(x^2 + 2xh + h^2) + x + h + 5 + 2x^2 - x - 5}{h}$$
$$= \frac{-2x^2 - 4xh - 2h^2 + x + h + 5 + 2x^2 - x - 5}{h}$$
$$= \frac{-4xh - 2h^2 + h}{h}$$
$$= \frac{h(-4x - 2h + 1)}{h}$$
$$= -4x - 2h + 1, \ h \neq 0$$

Pencil Problem #5

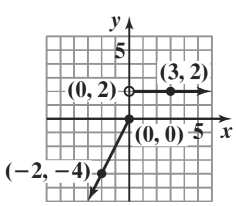
5. Find and simplify the difference quotient for $f(x) = x^2 - 4x + 3$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. decreasing on $(-\infty, -1)$; increasing on $(-1, \infty)$ *(1.3 #1)*

2. relative minimum of 0 at $x = -3$; relative maximum of 4 at $x = 0$; relative minimum of 0 at $x = 3$ *(1.3 #13)*

3a. odd *(1.3 #17)* **3b.** neither *(1.3 #19)* **3c.** even *(1.3 #21)*

4. 

$$f(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$$ *(1.3 #45)*

5. $2x + h - 4, \ h \neq 0$ *(1.3 #61)*

Section 1.4
Linear Functions and Slope

READ FOR LIFE!

Is there a relationship between literacy and child mortality?

As the percentage of adult females who are literate increases, does the mortality of children under age five decrease? Data from the United Nations indicates that this is, indeed, the case.

In this section of the textbook, you will be given a graph for which each point represents one country. You will use the concept of slope to see how much the mortality rate decreases for each 1% increase in the literacy rate of adult females in a country.

Objective #1: Calculate a line's slope.

✔ *Solved Problem #1*

1. Find the slope of the line passing through $(4,-2)$ and $(-1,5)$.

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$m=\frac{5-(-2)}{-1-4}$$

$$=\frac{7}{-5}$$

$$=-\frac{7}{5}$$

✎ *Pencil Problem #1* ✎

1. Find the slope of the line passing through $(-2,1)$ and $(2,2)$.

Objective #2: Write the point-slope form of the equation of a line.

✔ *Solved Problem #2*

2a. Write the point-slope form of the equation of the line with slope 6 that passes through the point $(2,-5)$. Then solve the equation for *y*.

Begin by finding the point-slope equation of a line.

$$y-y_1=m(x-x_1)$$

$$y-(-5)=6(x-2)$$

$$y+5=6(x-2)$$

Now solve this equation for *y*.

$$y+5=6(x-2)$$

$$y+5=6x-12$$

$$y=6x-17$$

✎ *Pencil Problem #2* ✎

2a. Write the point-slope form of the equation of the line with slope -3 that passes through the point $(-2,-3)$. Then solve the equation for *y*.

2b. A line passes through the points $(-2,-1)$ and $(-1,-6)$. Find the equation of the line in point-slope form and then solve the equation for y.

Begin by finding the slope: $m=\dfrac{-6-(-1)}{-1-(-2)}=\dfrac{-5}{1}=-5$

Using the slope and either point, find the point-slope equation of a line.

$$y-y_1=m(x-x_1) \quad \text{or} \quad y-y_1=m(x-x_1)$$
$$y-(-1)=-5(x-(-2)) \quad\quad y-(-6)=-5(x-(-1))$$
$$y+1=-5(x+2) \quad\quad y+6=-5(x+1)$$

To obtain slope-intercept form, solve the above equation for y:

$$y+1=-5(x+2) \quad \text{or} \quad y+6=-5(x+1)$$
$$y+1=-5x-10 \quad\quad y+6=-5x-5$$
$$y=-5x-11 \quad\quad y=-5x-11$$

2b. A line passes through the points $(-3,-1)$ and $(2,4)$. Find the equation of the line in point-slope form and then solve the equation for y.

Objective #3: Write and graph the slope-intercept form of the equation of a line.

✔ *Solved Problem #3*

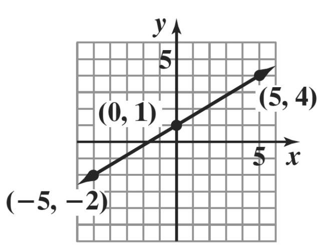
3. Graph: $f(x)=\dfrac{3}{5}x+1$

The y-intercept is 1, so plot the point $(0,1)$.

The slope is $m=\dfrac{3}{5}$.

Find another point by going up 3 units and to the right 5 units.

Use a straightedge to draw a line through the two points.



$f(x)=\dfrac{3}{5}x+1$

Pencil Problem #3

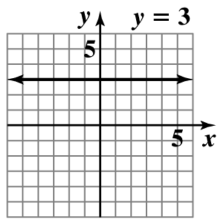
3. Graph: $f(x)=\dfrac{3}{4}x-2$

Objective #4: Graph horizontal or vertical lines.

✔ *Solved Problem #4*

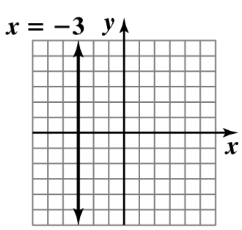
4a. Graph $y = 3$ in the rectangular coordinate system.

$y = 3$ is a horizontal line.



4b. Graph $x = -3$ in the rectangular coordinate system.

$x = -3$ is a vertical line.



✎ *Pencil Problem #4* ✐

4a. Graph $y = -2$ in the rectangular coordinate system.

4b. Graph $x = 5$ in the rectangular coordinate system.

Objective #5: Recognize and use the general form of a line's equation.

✔ Solved Problem #5

5. Find the slope and y-intercept of the line whose equation is $3x+6y-12=0$.

Solve for y.

$$3x+6y-12=0$$
$$6y=-3x+12$$
$$\frac{6y}{6}=\frac{-3x+12}{6}$$
$$y=\frac{-3}{6}x+\frac{12}{6}$$
$$y=-\frac{1}{2}x+2$$

The coefficient of x, $-\frac{1}{2}$, is the slope, and the constant term, 2, is the y-intercept.

✎ Pencil Problem #5

5. Find the slope and y-intercept of the line whose equation is $2x+3y-18=0$.

Objective #6: Use intercepts to graph a linear function in standard form.

✔ Solved Problem #6

6. Graph: $3x-2y-6=0$

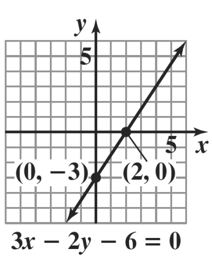
Find the x–intercept by setting $y = 0$.

$$3x-2y-6=0$$
$$3x-2(0)-6=0$$
$$3x=6$$
$$x=2$$

Find the y–intercept by setting $x = 0$.

$$3x-2y-6=0$$
$$3(0)-2y-6=0$$
$$-2y=6$$
$$y=-3$$

Plot the points and draw the line that passes through them.



✎ Pencil Problem #6

6. Graph: $6x-2y-12=0$

Objective #7: Model data with linear functions and make predictions.

✔ Solved Problem #7

7. The amount of carbon dioxide in the atmosphere, measured in parts per million, has been increasing as a result of the burning of oil and coal. The buildup of gases and particles is believed to trap heat and raise the planet's temperature. When the atmospheric concentration of carbon dioxide is 317 parts per million, the average global temperature is 57.04°F. When the atmospheric concentration of carbon dioxide is 354 parts per million, the average global temperature is 57.64°F.

Write a linear function that models average global temperature, $f(x)$, for an atmospheric concentration of carbon dioxide of x parts per million. Use the function to project the average global temperature when the atmospheric concentration of carbon dioxide is 600 parts per million.

Write the equation of the line through the points (317, 57.04) and (354, 57.64). First find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{57.64 - 57.04}{354 - 317} = \frac{0.6}{37} \approx 0.016$$

Use this slope and the point (317, 57.04) in the point-slope form.

$$y - y_1 = m(x - x_1)$$
$$y - 57.04 = 0.016(x - 317)$$
$$y - 57.04 = 0.016x - 5.072$$
$$y = 0.016x + 51.968$$

Using function notation and rounding the constant, we have

$$f(x) = 0.016x + 52.0$$

To predict the temperature when the atmospheric concentration of carbon dioxide is 600 parts per million, find $f(600)$.

$$f(600) = 0.016(600) + 52.0 = 61.6$$

The model predicts an average global temperature of 61.6°F when the atmospheric concentration of carbon dioxide is 600 parts per million.

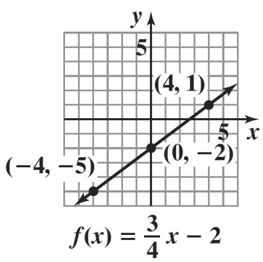
✎ Pencil Problem #7 ✐

7. When the literacy rate for adult females in a country is 0%, the infant mortality rate is 254 (per thousand). When the literacy rate for adult females is 60%, the infant mortality rate is 110. Write a linear function that models child mortality, $f(x)$, per thousand, for children under five in a country where x% of adult women are literate. Use the function to predict the child mortality rate in a country where 80% of adult females are literate.

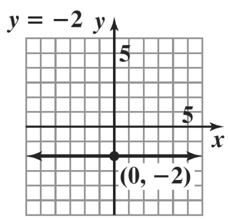
<u>Answers</u> <u>for Pencil Problems *(Textbook Exercise references in parentheses)*</u>:

1. $\frac{1}{4}$ *(1.4 #3)*

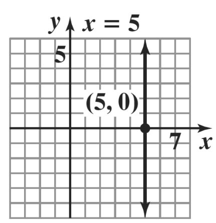
2a. $y+3=-3(x+2)$; $y=-3x-9$ *(1.4 #15)* **2b.** $y+1=1(x+3)$ or $y-4=1(x-2)$; $y=x+2$ *(1.4 #29)*

3. 

(1.4 #43)

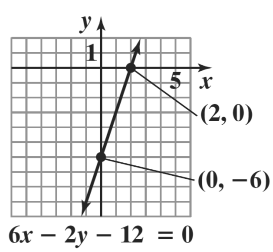
4a. 

(1.4 #49)

4b. 

(1.4 #52)

5. slope: $-\frac{2}{3}$; y-intercept: 6 *(1.4 #61)*

6. 

(1.4 #67)

7. $f(x)=-2.4x+254$; 62 per thousand *(1.4 #71)*

Section 1.5
More on Slope

Will They Ever Catch Up?

Many quantities, such as the number of men and the number of women living alone, are increasing over time. We can use slope to indicate how fast such quantities are growing on average.

If the slopes are the same, the quantities are growing at the same rate. However, if the slopes are different, then one quantity is growing faster than the other.
There were 9.0 million men and 14.0 million women living alone in 1990. Since then the number of men living alone has increased faster than the number of women living alone. If this trend continues, eventually, the number of men living alone will catch up to the number of women living alone.

Objective #1: Find slopes and equations of parallel and perpendicular lines.

✔ *Solved Problem #1*

1a. Write an equation of the line passing through $(-2,5)$ and parallel to the line whose equation is $y=3x+1$. Express the equation in point-slope form and slope-intercept form.

Since the line is parallel to $y=3x+1$, we know it will have slope $m=3$.

We are given that it passes through $(-2,5)$. We use the slope and point to write the equation in point-slope form.

$$y-y_1=m(x-x_1)$$
$$y-5=3(x-(-2))$$
$$y-5=3(x+2)$$

Point-Slope form: $y-5=3(x+2)$

Solve for y to obtain slope-intercept form.

$$y-5=3(x+2)$$
$$y-5=3x+6$$
$$y=3x+11$$
$$f(x)=3x+11$$

Slope-Intercept form: $y=3x+11$

✎ *Pencil Problem #1* ✐

1a. Write an equation of the line passing through $(-8,-10)$ and parallel to the line whose equation is $y=-4x+3$. Express the equation in point-slope form and slope-intercept form.

1b. Write an equation of the line passing through $(-2,-6)$ and perpendicular to the line whose equation is $x+3y-12=0$. Express the equation in point-slope form and general form.

First, find the slope of the line $x+3y-12=0$.
Solve the given equation for y to obtain slope-intercept form.

$$x+3y-12=0$$
$$3y=-x+12$$
$$y=-\frac{1}{3}x+4$$

Since the slope of the given line is $-\frac{1}{3}$, the slope of any line perpendicular to the given line is 3.

We use the slope of 3 and the point $(-2,-6)$ to write the equation in point-slope form. Then gather the variable and constant terms on one side with zero on the other side.

$$y-y_1=m(x-x_1)$$
$$y-(-6)=3(x-(-2))$$
$$y+6=3(x+2)$$
$$y+6=3x+6$$
$$0=3x-y \text{ or } 3x-y=0$$

1b. Write an equation of the line passing through $(4,-7)$ and perpendicular to the line whose equation is $x-2y-3=0$. Express the equation in point-slope form and general form.

Objective #2: Interpret slope as rate of change.

✔ *Solved Problem #2*

2. In 1990, there 9.0 million men living alone and in 2008, there were 14.7 million men living alone. Use the ordered pairs (1990, 9.0) and (2008, 14.7) to find the slope of the line through the points. Express the slope correct to two decimal places and describe what it represents.

$$m=\frac{\text{Change in } y}{\text{Change in } x}=\frac{14.7-9.0}{2008-1990}$$
$$=\frac{5.7}{18}\approx 0.32$$

The number of men living alone increased at an average rate of approximately 0.32 million men per year.

✎ *Pencil Problem #2* ✐

2. In 1994, 617 active-duty servicemembers were discharged under the “don’t ask, don’t tell” policy. In 1998, 1163 were discharged under the policy. Use the ordered pairs (1994, 617) and (1998, 1163) to find the slope of the line through the points. Express the slope correct to the nearest whole number and describe what it represents.

Objective #3: Find a function's average rate of change..

✔ *Solved Problem #3*

3a. Find the average rate of change of $f(x) = x^3$ from $x_1 = 0$ to $x_2 = 1$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0}$$
$$= \frac{1^3 - 0^3}{1}$$
$$= 1$$

The average rate of change is 1.

3b. Find the average rate of change of $f(x) = x^3$ from $x_1 = 1$ to $x_2 = 2$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(2) - f(1)}{2 - 1}$$
$$= \frac{2^3 - 1^3}{1}$$
$$= \frac{8 - 1}{1}$$
$$= 7$$

The average rate of change is 7.

3c. Find the average rate of change of $f(x) = x^3$ from $x_1 = -2$ to $x_2 = 0$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(0) - f(-2)}{0 - (-2)}$$
$$= \frac{0^3 - (-2)^3}{0 + 2}$$
$$= \frac{0 - (-8)}{2}$$
$$= \frac{8}{2}$$
$$= 4$$

The average rate of change is 4.

✎ *Pencil Problem #3* ✎

3a. Find the average rate of change of $f(x) = 3x$ from $x_1 = 0$ to $x_2 = 5$.

3b. Find the average rate of change of $f(x) = x^2 + 2x$ from $x_1 = 3$ to $x_2 = 5$

3c. Find the average rate of change of $f(x) = \sqrt{x}$ from $x_1 = 4$ to $x_2 = 9$.

3d. The distance, $s(t)$, in feet, traveled by a ball rolling down a ramp is given by the function $s(t) = 4t^2$, where t is the time, in seconds, after the ball is released. Find the ball's average velocity from $t_1 = 1$ second to $t_2 = 2$ seconds.

$$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{4 \cdot 2^2 - 4 \cdot 1^2}{1} = \frac{16 - 4}{1} = 12 \text{ ft/sec}$$

3d. The distance, $s(t)$, in feet, traveled by a ball rolling down a ramp is given by the function $s(t) = 10t^2$, where t is the time, in seconds, after the ball is released. Find the ball's average velocity from $t_1 = 3$ second to $t_2 = 4$ seconds.

3e. The distance, $s(t)$, in feet, traveled by a ball rolling down a ramp is given by the function $s(t) = 4t^2$, where t is the time, in seconds, after the ball is released. Find the ball's average velocity from $t_1 = 1$ second to $t_2 = 1.5$ seconds.

$$\frac{\Delta s}{\Delta t} = \frac{s(1.5) - s(1)}{1.5 - 1} = \frac{4 \cdot 1.5^2 - 4 \cdot 1^2}{0.5} = \frac{9 - 4}{0.5} = 10 \text{ ft/sec}$$

3e. The distance, $s(t)$, in feet, traveled by a ball rolling down a ramp is given by the function $s(t) = 10t^2$, where t is the time, in seconds, after the ball is released. Find the ball's average velocity from $t_1 = 3$ second to $t_2 = 3.5$ seconds.

<u>**Answers**</u> <u>for Pencil Problems</u> <u>*(Textbook Exercise references in parentheses)*</u>:

1a. Point-Slope form: $y + 10 = -4(x + 8)$, Slope-Intercept form: $y = -4x - 42$ *(1.5 #5)*

1b. Point-Slope form: $y + 7 = -2(x - 4)$, General form: $2x + y - 1 = 0$ *(1.5 #11)*

2. 137; There was an average increase of approximately 137 discharges per year. *(1.5 #27)*

3a. 3 *(1.5 #13)* **3b.** 10 *(1.5 #15)* **3c.** $\frac{1}{5}$ *(1.5 #17)*

3d. 70 ft/sec *(1.5 #19a)* **3e.** 60 ft/sec *(1.5 #19b)*

Section 1.6
Transformations of Functions

Movies and Mathematics

Have you ever seen special effects in a movie where a person or object is continuously transformed into something different? This is called morphing.
In mathematics, we can use transformations of a known graph to graph a function with a similar equation. This is achieved through horizontal and vertical shifts, reflections, and stretching and shrinking of the known graph.

Objective #1: Recognize graphs of common functions.

✔ *Solved Problem #1*

1. True or false: The graphs of the standard quadratic function $f(x) = x^2$ and the absolute value function $g(x) = |x|$ have the same type of symmetry.

True. Both functions are even and their graphs are symmetric with respect to the *y*-axis.

✎ *Pencil Problem #1* ✎

1. True or false: The graphs of the identity function $f(x) = x$ and the standard cubic function $g(x) = x^3$ have the same type of symmetry.

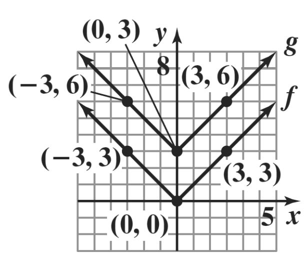
Objective #2: Use vertical shifts to graph functions.

✔ *Solved Problem #2*

2. Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = |x| + 3$.

The graph of *g* is the graph of *f* shifted vertically up by 3 units. Add 3 to each *y*-coordinate.

Since the points (–3, 3), (0, 0), and (3, 3) are on the graph of *f*, the points (–3, 6), (0, 3), and (3, 6) are on the graph of *g*.



✎ *Pencil Problem #2* ✎

2. Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 - 2$.

Objective #3: Use horizontal shifts to graph functions.

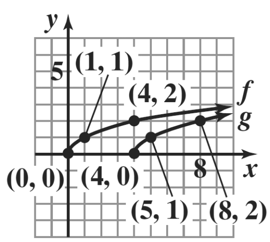
✔ *Solved Problem #3*

3. Use the graph of $f(x)=\sqrt{x}$ to obtain the graph of $g(x)=\sqrt{x-4}$.

$g(x)=\sqrt{x-4}=f(x-4)$

The graph of *g* is the graph of *f* shifted horizontally to the right by 4 units. Add 4 to each *x*-coordinate.

Since the points (0, 0), (1, 1), and (4, 2) are on the graph of *f*, the points (4, 0), (5, 1), and (8, 2) are on the graph of *g*.



✎ *Pencil Problem #3* ✎

3. Use the graph of $f(x)=|x|$ to obtain the graph of $g(x)=|x+4|$.

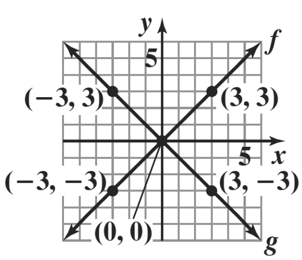
Objective #4: Use reflections to graph functions.

✔ *Solved Problem #4*

4a. Use the graph of $f(x)=|x|$ to obtain the graph of $g(x)=-|x|$.

The graph of *g* is a reflection of the graph of *f* about the *x*-axis because $g(x)=-f(x)$. Replace each *y*-coordinate with its opposite.

Since the points (–3, 3), (0, 0), and (3, 3) are on the graph of *f*, the points (–3, –3), (0, 0), and (3, –3) are on the graph of *g*.



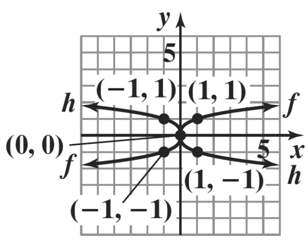
✎ *Pencil Problem #4* ✎

4a. Use the graph of $f(x)=x^3$ to obtain the graph of $h(x)=-x^3$.

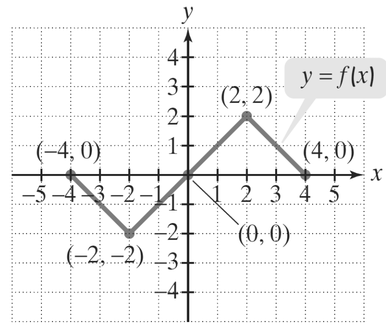
4b. Use the graph of $f(x)=\sqrt[3]{x}$ to obtain the graph of $h(x)=\sqrt[3]{-x}$.

The graph of h is a reflection of the graph of f about the y-axis because $h(x)=f(-x)$. Replace each x-coordinate with its opposite.

Since the points (–1, –1), (0, 0), and (1, 1) are on the graph of f, the points (1, –1), (0, 0), and (–1, 1) are on the graph of h.



4b. Use the graph of f shown below to obtain the graph of $g(x)=f(-x)$.



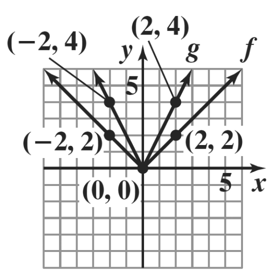
Objective #5: Use vertical stretching and shrinking to graph functions.

✔ *Solved Problem #5*

5. Use the graph of $f(x)=|x|$ to obtain the graph of $g(x)=2|x|$.

The graph of g is obtained by vertically stretching the graph of f because $g(x)=2f(x)$. Multiply each y-coordinate by 2.

Since the points (–2, 2), (0, 0), and (2, 2) are on the graph of f, the points (–2, 4), (0, 0), and (2, 4) are on the graph of g.



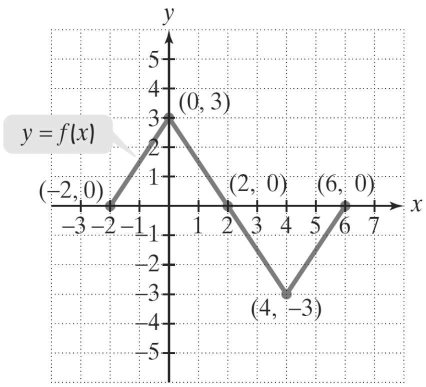
✎ *Pencil Problem #5* ✎

5. Use the graph of $f(x)=x^3$ to obtain the graph of $h(x)=\frac{1}{2}x^3$.

Objective #6: Use horizontal stretching and shrinking to graph functions.

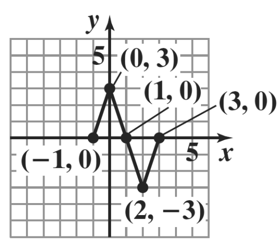
✔ *Solved Problem #6*

6. Use the graph of *f* shown below to graph each function.



6a. $g(x) = f(2x)$

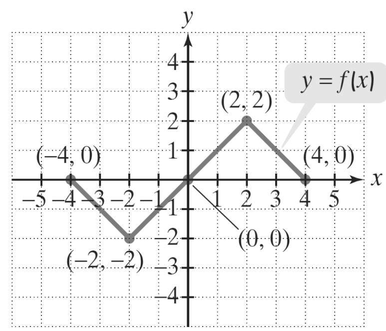
Divide the *x*-coordinate of each point on the graph of *f* by 2. The points (–1, 0), (0, 3), (1, 0), (2, –3), and (3, 0) are on the graph of *g*.



$g(x) = f(2x)$

✎ *Pencil Problem #6* ✎

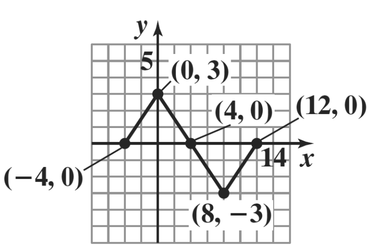
6. Use the graph of *f* shown below to graph each function.



6a. $g(x) = f(2x)$

6b. $h(x) = f(\frac{1}{2}x)$

Multiply the *x*-coordinate of each point on the graph of *f* by 2. The points (–4, 0), (0, 3), (4, 0), (8, –3), and (12, 0) are on the graph of *h*.



$h(x) = f\left(\frac{1}{2}x\right)$

6b. $g(x) = f(\frac{1}{2}x)$

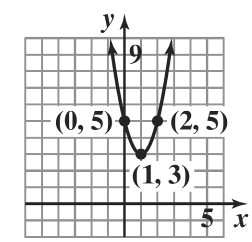
Objective #7: Graph functions involving a sequence of transformations.

✔ *Solved Problem #7*

7. Use the graph of $f(x) = x^2$ to graph $g(x) = 2(x-1)^2 + 3.$

The graph of *g* is the graph of *f* horizontally shifted to the right 1 unit, vertically stretched by a factor of 2, and vertically shifted up 3 units. Beginning with a point on the graph of *f*, add 1 to each *x*-coordinate, then multiply each *y*-coordinate by 2, and finally add 3 to each *y*-coordinate.

$(-1, 1) \to (0, 1) \to (0, 2) \to (0, 5)$
$(0, 0) \to (1, 0) \to (1, 0) \to (1, 3)$
$(1, 1) \to (2, 1) \to (2, 2) \to (2, 5)$



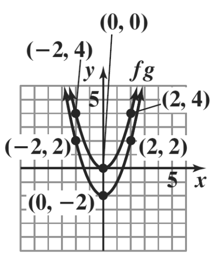
$g(x) = 2(x-1)^2 + 3$

✎ *Pencil Problem #7* ✎

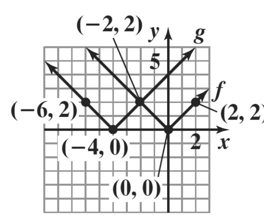
7. Use the graph of $f(x) = x^3$ to graph $h(x) = \frac{1}{2}(x-3)^2 - 2.$

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

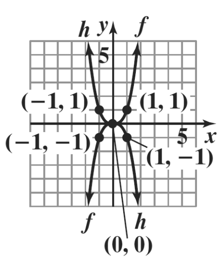
1. True *(1.6 #99)*

2. 

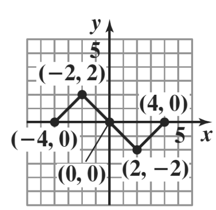
(1.6 #53)

3. 

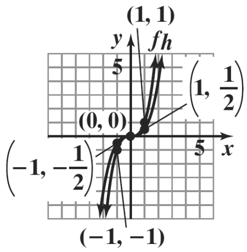
(1.6 #83)

4a. 

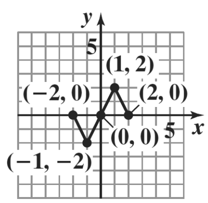
(1.6 #99)

4b. 

$g(x) = f(-x)$ *(1.6 #24)*

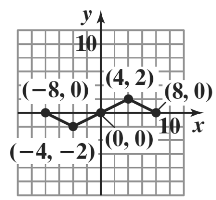
5. 

(1.6 #101)

6a. 

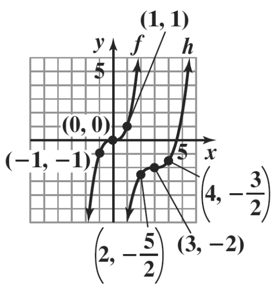
$g(x) = f(2x)$

(1.6 #29)

6b. 

$g(x) = f\left(\frac{1}{2}x\right)$

(1.6 #30)

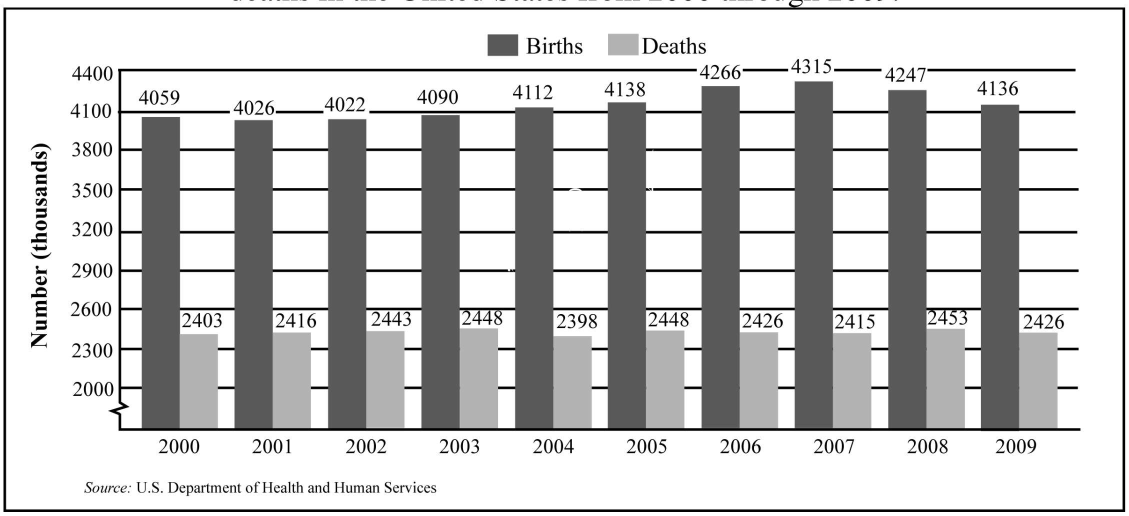
7. 

(1.6 #105)

Section 1.7
Combinations of Functions; Composite Functions

We're Born. We Die.

The figure below quantifies these statements by showing the number of births and deaths in the United States from 2000 through 2009.



In this section, we look at these data from the perspective of functions. By considering the yearly change in the U.S. population, you will see that functions can be subtracted using procedures that will remind you of combining algebraic expressions.

Objective #1: Find the domain of a function.

✔ *Solved Problem #1*

1a. Find the domain of $f(x) = x^2 + 3x - 17$.

The function contains neither division nor an even root. It is defined for all real numbers. The domain is $(-\infty, \infty)$.

1b. Find the domain of $j(x) = \dfrac{5x}{\sqrt{24-3x}}$.

The function contains both an even root and division. The expression under the radical must be nonnegative and the denominator cannot equal 0. Thus, $24 - 3x$ must be greater than 0.

$$24 - 3x > 0$$
$$24 > 3x$$
$$8 > x \quad \text{or} \quad x < 8$$

The domain is $(-\infty, 8)$.

✎ *Pencil Problem #1* ✎

1a. Find the domain of $f(x) = 3(x-4)$.

1b. Find the domain of $g(x) = \dfrac{\sqrt{x-2}}{x-5}$.

Objective #2: Combine functions using the algebra of functions, specifying domains.

✔ *Solved Problem #2*

2. Let $f(x) = x - 5$ and $g(x) = x^2 - 1$. Find each function and determine its domain.

2a. $(f+g)(x)$

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= (x-5) + (x^2-1) \\ &= x^2 + x - 6\end{aligned}$$

The domain is $(-\infty, \infty)$.

2b. $(f-g)(x)$

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) \\ &= (x-5) - (x^2-1) \\ &= x - 5 - x^2 + 1 \\ &= -x^2 + x - 4\end{aligned}$$

The domain is $(-\infty, \infty)$.

2c. $(fg)(x)$

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x) \\ &= (x-5)(x^2-1) \\ &= x^3 - 5x^2 - x + 5\end{aligned}$$

The domain is $(-\infty, \infty)$.

2d. $\left(\frac{f}{g}\right)(x)$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x-5}{x^2-1}$$

The function contains division; it is undefined when $x^2 - 1 = 0$ or $x^2 = 1$ or $x = \pm 1$. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

✎ *Pencil Problem #2* ✐

2. Let $f(x) = 2x^2 - x - 3$ and $g(x) = x + 1$. Find each function and determine its domain.

2a. $(f+g)(x)$

2b. $(f-g)(x)$

2c. $(fg)(x)$

2d. $\left(\frac{f}{g}\right)(x)$

Objective #3: Form composite functions.

✔ *Solved Problem #3*

3. Given $f(x)=5x+6$ and $g(x)=2x^2-x-1$, find each of the following.

3a. $(f\circ g)(x)$

$$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=5g(x)+6\\&=5(2x^2-x-1)+6\\&=10x^2-5x-5+6\\&=10x^2-5x+1\end{aligned}$$

3b. $(g\circ f)(x)$

$$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=2[f(x)]^2-f(x)-1\\&=2(5x+6)^2-(5x+6)-1\\&=2(25x^2+60x+36)-5x-6-1\\&=50x^2+120x+72-5x-7\\&=50x^2+115x+65\end{aligned}$$

3c. $(f\circ g)(-1)$

$$\begin{aligned}(f\circ g)(-1)&=10(-1)^2-5(-1)+1\\&=10+5+1\\&=16\end{aligned}$$

✎ *Pencil Problem #3* ✎

3. Given $f(x)=4x-3$ and $g(x)=5x^2-2$, find each of the following.

3a. $(f\circ g)(x)$

3b. $(g\circ f)(x)$

3c. $(f\circ g)(2)$

Objective #4: Determine domains for composite functions.

✔ *Solved Problem #4*

4. Given $f(x)=\dfrac{4}{x+2}$ and $g(x)=\dfrac{1}{x}$, find each of the following.

4a. $(f\circ g)(x)$

$$(f\circ g)(x)=\frac{4}{g(x)+2}=\frac{4}{\frac{1}{x}+2}\cdot\frac{x}{x}=\frac{4x}{1+2x}$$

✎ *Pencil Problem #4* ✎

4. Given $f(x)=\dfrac{2}{x+3}$ and $g(x)=\dfrac{1}{x}$, find each of the following.

4a. $(f\circ g)(x)$

4b. The domain of $f \circ g$

The function g is undefined when $x = 0$, so 0 is not in the domain of $f \circ g$. The function f is undefined for $x = -2$, so any values of x for which $g(x) = -2$ are not in the domain of $f \circ g$. Solving $\frac{1}{x} = -2$, we find that $x = -\frac{1}{2}$.

The domain is $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right) \cup (0, \infty)$.

4b. The domain of $f \circ g$

Objective #5: Write functions as compositions.

✔ *Solved Problem #5*

5. Express $h(x) = \sqrt{x^2 + 5}$ as the composition of two functions.

A natural way to write h as the composition of two functions is to take the square root of $g(x) = x^2 + 5$.

Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$.

Then $(f \circ g)(x) = \sqrt{g(x)} = \sqrt{x^2 + 5} = h(x)$.

✎ *Pencil Problem #5* ✎

5. Express $h(x) = (3x - 1)^4$ as the composition of two functions.

<u>**Answers** for Pencil Problems *(Textbook Exercise references in parentheses)*</u>:

1a. $(-\infty, \infty)$ *(1.7 #1)* **1b.** $[2, 5) \cup (5, \infty)$ *(1.7 #27)*

2a. $(f + g)(x) = 2x^2 - 2$; domain: $(-\infty, \infty)$ *(1.7 #35)*

2b. $(f - g)(x) = 2x^2 - 2x - 4$; domain: $(-\infty, \infty)$ *(1.7 #35)*

2c. $(fg)(x) = 2x^3 + x^2 - 4x - 3$; domain: $(-\infty, \infty)$ *(1.7 #35)*

2d. $\left(\frac{f}{g}\right)(x) = \frac{2x^2 - x - 3}{x + 1}$; domain: $(-\infty, -1) \cup (1, \infty)$ *(1.7 #35)*

3a. $(f \circ g)(x) = 20x^2 - 11$ *(1.7 #55a)* **3b.** $(g \circ f)(x) = 80x^2 - 120x + 43$ *(1.7 #55 b)* **3c.** 69 *(1.7 #55c)*

4a. $(f \circ g)(x) = \frac{2x}{1 + 3x}$ *(1.7 #67a)* **4b.** $\left(-\infty, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, 0\right) \cup (0, \infty)$ *(1.7 #67b)*

5. Let $f(x) = x^4$ and $g(x) = 3x - 1$. Then $(f \circ g)(x) = h(x)$. *(1.7 #75)*

Section 1.8
Inverse Functions

Hey! That's My Birthday Too !

What is the probability that two people in the same room share a birthday?

It might be higher than you think.

In this section we will explore the graph of the function that represents this probability.

Objective #1: Verify inverse functions.

✔ Solved Problem #1

1. Show that each function is the inverse of the other:
$f(x) = 4x - 7$ and $g(x) = \frac{x+7}{4}$.

First, show that $f\left(g(x)\right) = x$.

$$\begin{aligned} f\left(g(x)\right) &= 4\left(\frac{x+7}{4}\right) - 7 \\ &= x + 7 - 7 \\ &= x \end{aligned}$$

Next, show that $g\left(f(x)\right) = x$.

$$\begin{aligned} g\left(f(x)\right) &= \frac{(4x-7)+7}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

✎ Pencil Problem #1 ✎

1. Determine whether $f(x) = \frac{3}{x-4}$ and $g(x) = \frac{3}{x} + 4$ are inverses of each other.

Objective #2: Find the inverse of a function.

✔ Solved Problem #2

2a. Find the inverse of $f(x) = 2x + 7$.

Replace $f(x)$ with y.

$y = 2x + 7$

Interchange x and y and solve for y.

$$\begin{aligned} x &= 2y + 7 \\ x - 7 &= 2y \\ \frac{x-7}{2} &= y \end{aligned}$$

Replace y with $f^{-1}(x)$.

$f^{-1}(x) = \frac{x-7}{2}$

✎ Pencil Problem #2 ✎

2a. Find the inverse of $f(x) = x + 3$.

2b. Find the inverse of $f(x) = 4x^3 - 1$.

Replace *f*(*x*) with *y*.

$y = 4x^3 - 1$

Interchange *x* and *y* and solve for *y*.

$$x = 4y^3 - 1$$
$$x + 1 = 4y^3$$
$$\frac{x+1}{4} = y^3$$
$$\sqrt[3]{\frac{x+1}{4}} = y$$

Replace *y* with $f^{-1}(x)$.

$f^{-1}(x) = \sqrt[3]{\frac{x+1}{4}}$

2b. Find the inverse of $f(x) = (x+2)^3$.

2c. Find the inverse of $f(x) = \frac{3}{x} - 1$.

Replace *f*(*x*) with *y*.

$y = \frac{3}{x} - 1$

Interchange *x* and *y* and solve for *y*.

$$x = \frac{3}{y} - 1$$
$$xy = \left(\frac{3}{y} - 1\right)y$$
$$xy = \frac{3}{y} \cdot y - 1 \cdot y$$
$$xy = 3 - y$$
$$xy + y = 3$$
$$y(x+1) = 3$$
$$\frac{y(x+1)}{x+1} = \frac{3}{x+1}$$
$$y = \frac{3}{x+1}$$

Replace *y* with $f^{-1}(x)$.

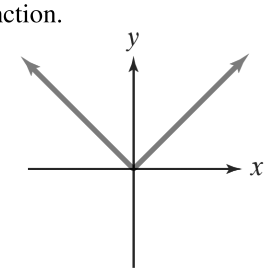
$f^{-1}(x) = \frac{3}{x+1}$

2c. Find the inverse of $f(x) = \frac{7}{x} - 3$.

Objective #3: Use the horizontal line test to determine if a function has an inverse function.

✔ *Solved Problem #3*

3. Use the horizontal line test to determine if the following graph represents a function that has an inverse function.

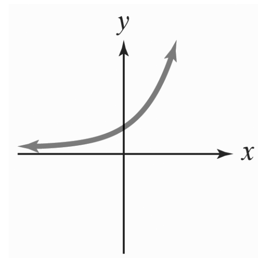


Since a horizontal line can be drawn that intersects the graph more than once, it fails the horizontal line test.

Thus, this graph does not represent a function that has an inverse function.

✎ *Pencil Problem #3* ✎

3. Use the horizontal line test to determine if the following graph represents a function that has an inverse function.



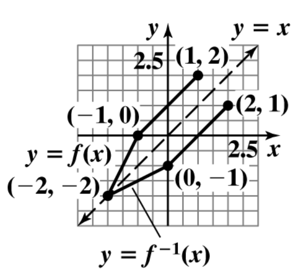
Objective #4: Use the graph of a one-to-one function to graph its inverse function.

✔ *Solved Problem #4*

4. The graph of function *f* consists of two line segments, one segment from $(-2,-2)$ to $(-1,0)$, and a second segment from $(-1,0)$ to $(1,2)$. Graph *f* and use the graph to draw the graph of its inverse function.

Since *f* has a line segment from $(-2,-2)$ to $(-1,0)$, then f^{-1} has a line segment from $(-2,-2)$ to $(0,-1)$.

Since *f* has a line segment from $(-1,0)$ to $(1,2)$, then f^{-1} has a line segment from $(0,-1)$ to $(2,1)$.



✎ *Pencil Problem #4* ✎

4. The graph of a linear function *f* contains the points $(0,-4)$, $(2,0)$, $(3,2)$, and $(4,4)$. Draw the graph of the inverse function.

Objective #5: Find the inverse of a function and graph both functions on the same axes.

✔ Solved Problem #5

5. Find the inverse of $f(x) = x^2 + 1$ if $x \geq 0$. Graph f and f^{-1} in the same rectangular coordinate system.

Restricted to $x \geq 0$, the function $f(x) = x^2 + 1$ has an inverse. The graph of f is the right half of the graph of $y = x^2$ shifted up 1 unit.

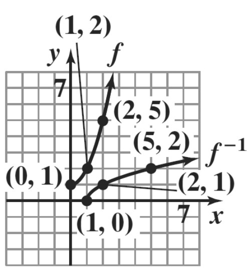
Replace $f(x)$ with y: $y = x^2 + 1$.

Interchange x and y and solve for y. Since the values of x are nonnegative in the original function, the values of y must be nonnegative in the inverse function. We choose the positive square root in the third step below.

$$x = y^2 + 1$$
$$x - 1 = y^2$$
$$\sqrt{x-1} = y$$

Replace y with $f^{-1}(x)$: $f^{-1}(x) = \sqrt{x-1}$.

The graph of f^{-1} is the graph of the square root function shifted 1 unit to the left. The graph of f^{-1} is also the reflection of the graph of f about the line $y = x$.



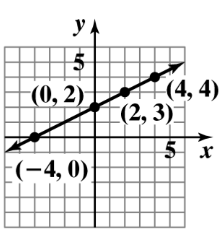
✎ Pencil Problem #5 ✎

5. Find the inverse of $f(x) = (x-1)^2$ if $x \leq 1$. Graph f and f^{-1} in the same rectangular coordinate system.

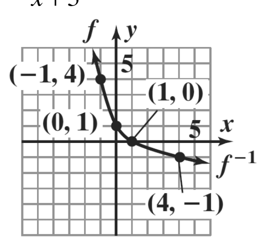
Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1. The functions are inverses of each other. *(1.8 #7)* **2a.** $f^{-1}(x) = x - 3$ *(1.8 #11)*

2b. $f^{-1}(x) = \sqrt[3]{x} - 2$ *(1.8 #19)* **2c.** $f^{-1}(x) = \frac{7}{x+3}$ *(1.8 #25)* **3.** has inverse function *(1.8 #33)*

4.


(1.8 #35)

5.


(1.8 #43)

Section 1.9
Distance and Midpoint Formulas; Circles

Round and Round !

In 1893, George Washington Gale Ferris, Jr. designed and built the first Ferris wheel as the centerpiece for the World's Columbian Exposition in Chicago.

The rectangular coordinate system gives us a unique way of knowing a circle. It enables us to translate a circle's geometric definition into an algebraic equation. In this section, we will learn, and then apply, these algebraic techniques.

Objective #1: Find the distance between two points.

✔ *Solved Problem #1*

1. Find the distance between $(-1,-3)$ and $(2,3)$. Express the answer in simplified radical form and then round to two decimal places.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d=\sqrt{(2-(-1))^2+(3-(-3))^2}$$
$$=\sqrt{3^2+6^2}$$
$$=\sqrt{9+36}$$
$$=\sqrt{45}$$
$$=3\sqrt{5}$$
$$\approx 6.71 \text{ units}$$

✎ *Pencil Problem #1* ✎

1. Find the distance between $(4,1)$ and $(6,3)$. Express the answer in simplified radical form and then round to two decimal places.

Objective #2: Find the midpoint of a line segment.

✔ *Solved Problem #2*

2. Find the midpoint of the line segment with endpoints $(1,2)$ and $(7,-3)$.

$$\text{Midpoint}=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$
$$=\left(\frac{1+7}{2},\frac{2+(-3)}{2}\right)$$
$$=\left(\frac{8}{2},\frac{-1}{2}\right)$$
$$=\left(4,-\frac{1}{2}\right)$$

✎ *Pencil Problem #2* ✎

2. Find the midpoint of the line segment with endpoints $(6,8)$ and $(2,4)$.

Objective #3: Write the standard form of a circle's equation.

✔ ***Solved Problem #3***

3a. Write the standard form of the equation of the circle with center $(0,0)$ and radius 4.

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-0)^2+(y-0)^2=4^2$$
$$x^2+y^2=16$$

✎ ***Pencil Problem #3*** ✎

3a. Write the standard form of the equation of the circle with center $(0,0)$ and radius 7.

✔ ***Solved Problem #3***

3b. Write the standard form of the equation of the circle with center $(0,-6)$ and radius 10.

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-0)^2+(y-(-6))^2=10^2$$
$$x^2+(y+6)^2=100$$

✎ ***Pencil Problem #3*** ✎

3b. Write the standard form of the equation of the circle with center $(-1,4)$ and radius 2.

Objective #4: Give the center and radius of a circle whose equation is in standard form.

✔ ***Solved Problem #4***

4a. Find the center and radius of the circle whose equation is $(x+3)^2+(y-1)^2=4$.

$$(x+3)^2+(y-1)^2=4$$
$$(x-(-3))^2+(y-1)^2=2^2$$

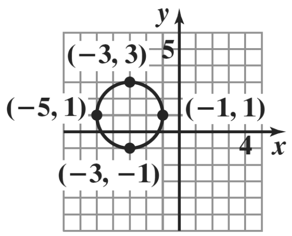
The center is $(-3,1)$ and the radius is 2 units.

✎ ***Pencil Problem #4*** ✎

4a. Find the center and radius of the circle whose equation is $(x-3)^2+(y-1)^2=36$.

4b. Graph the equation in Solved Problem 4a.

Plot points 2 units above and below and to the left and right of the center, (–3,1). Draw a circle through these points.



4b. Graph the equation in Pencil Problem 4a.

4c. Use the graph in Solved Problem 4b to identify the relation's domain and range.

The leftmost point on the circle has an x-coordinate of –5, and the rightmost point has an x-coordinate of –1. The domain is [–5, –1].

The lowest point on the graph has a y-coordinate of –1, and the highest point on the graph has a y-coordinate of 3. The range is [–1, 3].

4c. Use the graph in Pencil Problem 4b to identify the relation's domain and range.

Objective #5: Convert the general form of a circle's equation to standard form.

✔ *Solved Problem #5*

5. Write in standard form and graph:

$$x^2+y^2+4x-4y-1=0$$

$$x^2+y^2+4x-4y-1=0$$

$$\left(x^2+4x\quad\right)+\left(y^2-4y\quad\right)=1$$

Complete the squares.

For x: $\left(\frac{b}{2}\right)^2=\left(\frac{4}{2}\right)^2=(2)^2=4$

For y: $\left(\frac{b}{2}\right)^2=\left(\frac{-4}{2}\right)^2=(-2)^2=4$

✎ *Pencil Problem #5* ✎

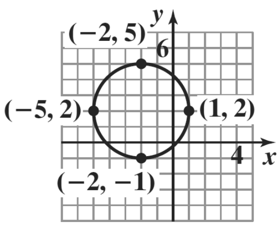
5. Write in standard form and graph:

$$x^2+y^2+8x-2y-8=0$$

Add these values to both sides of the equation.

$$x^2+y^2+4x-4y-1=0$$
$$\left(x^2+4x\quad\right)+\left(y^2-4y\quad\right)=1$$
$$\left(x^2+4x+4\right)+\left(y^2-4y+4\right)=1+4+4$$
$$(x+2)^2+(y-2)^2=9$$
$$(x-(-2))^2+(y-2)^2=3^2$$

The center is $(-2,2)$ and the radius is 3 units.



$x^2+y^2+4x-4y-1=0$

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

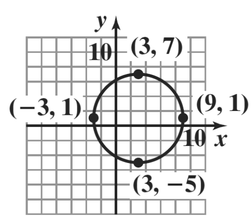
1. $2\sqrt{2}\approx 2.83$ *(1.9 #3)* **2.** $(4,6)$ *(1. 9#19)*

3a. $x^2+y^2=49$ *(1.9 #31)*

3b. $(x+1)^2+(y-4)^2=4$ *(1.9 #35)*

4a. center: $(3,1)$; radius: 6 units

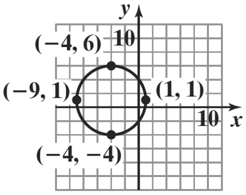
4b.



$(x-3)^2+(y-1)^2=36$ *(1.9 #43)*

4c. domain: $[-3,9]$; range: $[-5,7]$ *(1.9 #43)*

5. $(x+4)^2+(y-1)^2=25$;



$x^2+y^2+8x-2y-8=0$ *(1.9 #57)*

Section 1.10
Modeling with Functions

The Mathematics of Soda?

Although we have concerns about the effects of soft drinks on your health, we do want to explore the connection between mathematics and the size and shape of your soda can. Given that the can is supposed to hold 12 ounces of soda, how can we determine the dimensions of the can and its surface area?

In this section, you will learn how to express the surface area of a can with a fixed volume as a function of its radius, In calculus, you will take the problem a step further and find the radius that minimizes the surface area of the soda can.

Objective #1: Construct functions from verbal conditions.

✔ *Solved Problem #1*

1a. You are choosing between two texting plans. Plan A has a monthly fee of \$15 with a charge of \$0.08 per text. Plan B has a monthly fee of \$3 with a charge of \$0.12 per text. Express the monthly cost for each plan as a function of the numbers of text messages in a month, x. For how many text messages will the costs for the two plans be the same?

We start with plan A. There is a fee of \$15 plus a charge of \$0.08 per text for x text messages. Multiply the number of texts, x, by the cost per text, \$0.08, and add to the monthly fee, \$15.

$f(x) = 15 + 0.08x$ or $f(x) = 0.08x + 15$

For plan B, there is a fee of \$3 plus a charge of \$0.12 per text for x text messages. Multiply the number of texts, x, by the cost per text, \$0.12, and add to the monthly fee, \$3.

$g(x) = 3 + 0.12x$ or $g(x) = 0.12x + 3$

To find when the costs of the two plans are the same, we solve $f(x) = g(x)$.

$$\begin{aligned} f(x) &= g(x) \\ 0.08x + 15 &= 0.12x + 3 \\ 15 &= 0.04x + 3 \\ 12 &= 0.04x \\ 300 &= x \end{aligned}$$

The two plans have the same cost for 300 text messages. To check this, note that the cost of plan A for 300 text messages is

$f(300) = 0.08(300) + 15 = 24 + 15 = \$39,$

and the cost of plan B is

$g(300) = 0.12(300) + 3 = 36 + 3 = \$39.$

✎ *Pencil Problem #1* ✐

1a. You are choosing between two plans at a discount warehouse. Plan A has an annual membership of \$100 and you pay 80% of the manufacturer's recommended list price. Plan B has an annual membership of \$40 and you pay 90% of the manufacturer's recommended list price. Express the total yearly amount paid to the warehouse for each plan as a function of the dollars of merchandise purchased, x. For how many dollars of merchandise will the amount paid under the two plans be the same?

1b. On a certain route, an airline carries 8000 passengers per month, each paying \$100. For each \$1 increase in ticket price, the airline will lose 100 passengers. Express the number of passengers per month, *N*, as a function of the ticket price, *x*. Then express the monthly revenue for the route, *R*, as a function of the ticket price, *x*.

If the new ticket price is *x* dollars, then the amount the ticket price has increased over the original price, \$100, is $x - 100$.

For each \$1 increase, the airline will lose 100 passengers. So for an increase of $(x - 100)$ dollars, the airline will lose $100(x - 100)$ passengers.

The airline now has 8000 passengers. If it loses $100(x - 100)$ passengers, it will have $8000 - 100(x - 100)$ passengers.

$$\begin{aligned} N(x) &= 8000 - 100(x - 100) \\ &= 8000 - 100x + 10{,}000 \\ &= -100x + 18{,}000 \end{aligned}$$

So, $N(x) = -100x + 18{,}000$ models the number of passengers per month.

Since the ticket price is *x* dollars and there are $N(x)$ passengers, multiply $N(x)$ by *x* to find the revenue.

$$\begin{aligned} R(x) &= (-100x + 18{,}000) \cdot x \\ &= -100x^2 + 18{,}000x \end{aligned}$$

So, $R(x) = -100x^2 + 18{,}000x$ models the monthly revenue for the route.

1b. With a ticket price of \$20, the average attendance at football games is 30,000. For each \$1 increase in ticket price, attendance decreases by 500. Express the average attendance at a football game, *N*, as a function of the ticket price, *x*. Then express the revenue from a football game, *R*, as a function of the ticket price, *x*.

Objective #2: Construct functions from formulas.

✔ *Solved Problem #2*

2a. You have 200 feet of fencing to enclose a rectangular garden. Express the area, A, of the garden as function of one of its dimensions, x.

Let x and y be the length and the width of the garden, respectively. Since the garden is rectangular, it area is then $A = xy$. However, we want to express the area in terms of x only. We need to substitute an expression for y in terms of x.

We know that the perimeter of the garden will be 200 feet, since this is the amount of fencing available. We can also express the perimeter in terms of its length, x, and width, y.

Perimeter = 200 feet
Perimeter = $2x + 2y$

Solve the equation $2x + 2y = 200$ for y.

$$2x + 2y = 200$$
$$2y = 200 - 2x$$
$$\frac{2y}{2} = \frac{200 - 2x}{2}$$
$$y = 100 - x$$

Now replace y with $100 - x$ in $A = xy$ and write the result as a function.

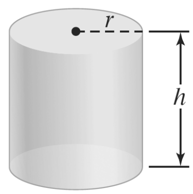
$$A(x) = x(100 - x) \text{ or } A(x) = 100x - x^2$$

✎ *Pencil Problem #2* ✎

2a. You have 800 feet of fencing to enclose a rectangular field. Express the area, A, of the field as function of one of its dimensions, x.

2b. A cylindrical can is to hold 1 liter, or 1000 cubic centimeters, of oil. Express the surface area of the can, *A*, in square centimeters, as a function of its radius, *r*, in centimeters.

$$A = 2\pi r^2 + 2\pi rh$$



The surface area of a cylindrical can is given by the formula $A = 2\pi r^2 + 2\pi rh$, where *r* is the radius and *h* is the height. We need to express this in terms of *r* only. We need to replace *h* with an expression in terms of *r*.

The volume of the can is 1000 cubic centimeters, and volume is given by the formula $V = \pi r^2 h$, where *r* is the radius and *h* is the height. Replace *V* with 1000 in the formula and solve for *h*.

$$V = \pi r^2 h$$
$$1000 = \pi r^2 h$$
$$\frac{1000}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$
$$\frac{1000}{\pi r^2} = h$$

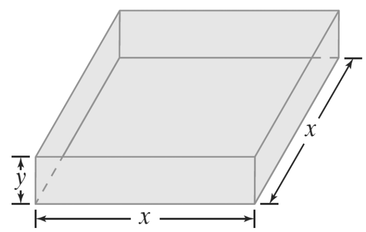
Now replace *h* in the surface area formula with $\frac{1000}{\pi r^2}$.

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r} \end{aligned}$$

Now write as a function.

$$A(r) = 2\pi r^2 + \frac{2000}{r}$$

2b. The figure shows an open box with a square base. The box is to have a volume of 10 cubic feet. Express the amount of material needed to construct the box, *A*, as a function of the length of its square base, *x*. [Hint: How is this problem similar to Solved Problem #2b? Both involve finding a function for the surface area of a figure when the volume is given.]



2c. Let $P = (x, y)$ be a point on the graph of $y = x^3$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

Since P is on the graph of $y = x^3$, the y-coordinate of P can be replaced by x^3. Thus,
$P = (x, y) = (x, x^3)$

Recall that the origin is the point with coordinates (0, 0). We use the distance formula to find the distance between (x, x^3) and (0, 0).

$$d = \sqrt{(x-0)^2 + (x^3 - 0)^2} = \sqrt{(x)^2 + (x^3)^2} = \sqrt{x^2 + x^6}$$

Now write as a function.
$d(x) = \sqrt{x^2 + x^6}$ or $d(x) = \sqrt{x^6 + x^2}$

2c. Let $P = (x, y)$ be a point on the graph of $y = x^2 - 4$. Express the distance, d, from P to the origin as a function of the point's x-coordinate.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. Plan A: $f(x) = 100 + 0.8x$; plan B: $g(x) = 40 + 0.9x$; \$600 of merchandise *(1.10 #7)*

1b. Attendance: $N(x) = -500x + 40,000$; revenue: $R(x) = -500x^2 + 40,000x$ *(1.10 #9)*

2a. $A(x) = x(400 - x)$ or $A(x) = 400x - x^2$ *(1.10 #21)*

2b. $A(x) = \frac{40}{x} + x^2$ *(1.10 #31)*

2c. $d(x) = \sqrt{x^4 - 7x^2 + 16}$ *(1.10 #39)*